The Complexity of Ferromagnetic Two-spin Systems with External Fields

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Abstract

We study the approximability of computing the partition function for ferromagnetic two-state spin systems. The remarkable algorithm by Jerrum and Sinclair showed that there is a fully polynomial-time randomized approximation scheme (FPRAS) for the special ferromagnetic Ising model with any given uniform external field. Later, Goldberg and Jerrum proved that it is #BIS-hard for Ising model if we allow inconsistent external fields on different nodes. In contrast to these two results, we prove that for any ferromagnetic two-state spin systems except the Ising model, there exists a threshold for external fields beyond which the problem is #BIS-hard, even if the external field is uniform.

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1 Introduction

Spin systems are well studied in statistical physics and applied probability. We focus on two-state spin systems in this paper. An instance of a spin system is described by a graph $G(V, E)$, where vertices are particles and edges indicate neighborhood relation among them. A configuration $\sigma : V \to \{0, 1\}$ assigns one of the two states to every vertex. The contribution of local interactions between adjacent vertices is quantified by a matrix $\mathbf{A} = \begin{bmatrix} A_{0,0} & A_{0,1} \\ A_{1,0} & A_{1,1} \end{bmatrix} = \begin{bmatrix} \beta & 1 \\ 1 & \gamma \end{bmatrix}$, where $\beta, \gamma \geq 0$. The contribution of vertices in different spin states is quantified by a vector $\mathbf{b} = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix} = \begin{bmatrix} \mu \\ 1 \end{bmatrix}$, where $\mu > 0$. This μ is also called the external field of the system, which indicates a priori preference of an isolate vertex. The *partition function* $Z_{(\beta,\gamma,\mu)}(G)$ of a spin system $G(V, E)$ is defined to be the following exponential summation:

$$Z_{(\beta,\gamma,\mu)}(G) \triangleq \sum_{\sigma \in \{0,1\}^V} \prod_{v \in V} b_{\sigma_v} \prod_{(u,v) \in E} A_{\sigma_u,\sigma_v}.$$

We call such a spin system parameterized by (β, γ, μ). If the parameters are clear from the context, we shall write $Z(G)$ for short. Although originated from statistical physics, the

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spin model is also accepted in computer science as a framework for counting problems. For example, with $\beta = 0$, $\gamma = 1$ and $\mu = 1$, $Z_{(\beta,\gamma,\mu)}(G)$ is the number of independent sets (or vertex covers) of the graph G.

Given a set of parameters (β, γ, μ), it is a computational problem to compute the partition function $Z_{(\beta,\gamma,\mu)}(G)$ with input G. We denote this computation problem as $\textsc{Spin}(\beta, \gamma, \mu)$ and want to characterize its computational complexity in terms of β, γ and μ. For *exact* computation, polynomial time algorithms only exist for the very restricted settings that $\beta\gamma = 1$ and $(\beta, \gamma) = (0, 0)$. For all other settings, the problem is known to be #P-hard [2]. Therefore, the main focus becomes to study its approximability. The notion of the fully polynomial-time approximation scheme (FPTAS) is defined as follows: A algorithm $\mathcal{A}$ is an FPTAS for $\textsc{Spin}(\beta, \gamma, \mu)$ if for any given parameter $\varepsilon > 0$, $\mathcal{A}$ outputs a number $\hat{Z}$ such that $Z(G)\exp(-\varepsilon) \leq \hat{Z} \leq Z(G)\exp(\varepsilon)$ and runs in time $poly(n, 1/\varepsilon)$, where n is the size of the graph G. The randomized relaxation of FPTAS is called fully polynomial-time randomized approximation scheme (FPRAS), which uses random bits and only requires the final output be within the required accuracy with high probability.

The spin systems (β, γ, μ) are classified into two families with distinct physical and computational properties: *ferromagnetic* systems ($\beta\gamma > 1$) and *anti-ferromagnetic* systems ($\beta\gamma < 1$). We shall denote the corresponding computation problems by $\textsc{Ferro}(\beta, \gamma, \mu)$ and $\textsc{Anti-Ferro}(\beta, \gamma, \mu)$ respectively, so as to emphasize which family these parameters belong to. Systems with $\beta\gamma = 1$ are degenerate and trivial both physically and computationally. As a result, we only study systems with $\beta\gamma \neq 1$.

Great progress has been made recently for approximately computing the partition function for anti-ferromagnetic two-spin systems: it admits an FPTAS up to the uniqueness threshold [20, 12, 13, 16], and is NP-hard to approximate in the non-uniqueness range [18, 6]. The uniqueness threshold is a phase transition boundary in physics. It is widely conjectured that the computational difficulty is related to the phase transition point in many problems; this is one of the very few examples where a rigorous proof is obtained.

For ferromagnetic systems, the picture is quite different. The uniqueness condition does not coincide with the transition of computational difficulty and it is not clear whether they have any relation. In a seminal paper [10], Jerrum and Sinclair gave an FPRAS for ferromagnetic Ising model $\beta = \gamma > 1$ with any external field μ. Thus, there is no transition of computational difficulty for ferromagnetic Ising model, which contrasts the situation for anti-ferromagnetic Ising model $\beta = \gamma < 1$. For general ferromagnetic spin systems with external field, the approximability is less clear. Since the Ising model ($\beta = \gamma$) is solved, in this paper, we focus on the case $\beta \neq \gamma$ and always assume $\beta < \gamma$ by symmetry. It is known that, an FPRAS exists for $\mu \leq \sqrt{\gamma/\beta}$ [9], by a reduction to Ising model.

On the other hand, a hardness result was obtained for Ising model with inconsistent external fields [7]. This is a generalization of the spin system where the external fields for vertices are no longer required to be uniform and are arbitrarily taken from a set $\mathcal{V}$. We use $\textsc{Spin}(\beta, \gamma, \mathcal{V})$ ($\textsc{Ferro}(\beta, \gamma, \mathcal{V})$ or $\textsc{Anti-Ferro}(\beta, \gamma, \mathcal{V})$) to denote this computation problem. It is proved that the Ising model with arbitrary external fields $\textsc{Ferro}(\beta, \beta, (0, +\infty))$ is #`BIS`-hard, namely the problem is at least as hard as counting independent sets on bipartite graphs (#`BIS`). #`BIS` is a problem of intermediate hardness and has been conjectured to admit no FPRAS [5]. The reduction used here is called approximation-preserving reduction as introduced in [4]: Let $A, B : \Sigma^* \to \mathbb{R}$ be two functions. An *approximation-preserving reduction* from A to B is a randomized polynomial-time algorithm that approximates A while using an oracle for B. We write $A \leq_{AP} B$ for short if an approximation-preserving reduction exists from A to B. To make use of the aforementioned #`BIS`-hardness result, one needs to use (or simulate) both arbitrarily small and large external fields. As $\beta < \gamma$, one can always simulate a arbitrarily small external field with a gadget. However, simulating a arbitrarily large external field is only possible when $\beta\mu + 1 > \mu + \gamma$, in which case a #BIS-hardness is immediate. If this is not the case, and in particular if $\beta \leq 1 < \gamma$, no hardness result was known for any bounded external fields. These systems have certain monotonicity property, so all external fields that can be simulated by gadgets are inherently bounded above. It was not even clear whether problems in this regime is hard. As our first result, we show that the problem is already hard as long as we allow sufficiently large (yet still bounded above) and vertex-dependent external fields.

▶ **Theorem 1.** *For any $\beta < \gamma$ with $\beta\gamma > 1$, there exists a bounded set $\mathcal{V}$ such that* $\textsc{Ferro}(\beta, \gamma, \mathcal{V})$ *is* #BIS*-hard.*

The main difficulty to establish the theorem is for the case of $\beta \leq 1$, for which we cannot simulate any external field larger than the upper bound of $\mathcal{V}$. We overcome this difficulty by making use of a recent beautiful result in [3]. Instead of starting with the independent set problem on arbitrary bipartite graphs, we start with a soft ($\beta\gamma > 0$) *anti*-ferromagnetic two-spin system on bipartite graphs of *bounded degree*. As a result, all the external fields needed for the reduction are bounded.

However, in the above reduction, we do need vertices to have different external fields to make the reduction go through. This gives a hardness result for $\textsc{Ferro}(\beta, \gamma, \mathcal{V})$ but not for $\textsc{Ferro}(\beta, \gamma, \mu)$ for a single μ. It is more interesting and intriguing (both physically and computationally) to understand the computational complexity of a uniform spin system (β, γ, μ) with the same external field μ on all vertices. As our main result of this paper, we prove #BIS-hardness for this uniform case for sufficiently large single external field μ. We prove that when μ is sufficiently large, we can realize by sufficient precision of all the external fields which is smaller than $\mu^*(\mu, \beta, \gamma)$, where $\mu^*(\mu, \beta, \gamma) < \mu$ is a function of μ, β and γ, and approaches infinity as μ goes to infinity. Then by choosing large enough μ and making use of Theorem 1, we obtain our main theorem.

▶ **Theorem 2.** *For any $\beta < \gamma$ with $\beta\gamma > 1$, there exist a μ_0 such that* $\textsc{Ferro}(\beta, \gamma, \mu)$ *is* #BIS*-hard for all $\mu \geq \mu_0$.*

Our main technical contribution is the construction of a family of gadgets to simulate a given target external field. We use a reverse idea of correlation decay to achieve this. Correlation decay is proved to be a very powerful technique to design FPTAS for counting problems (see for examples [20, 1, 13, 16, 14, 15]). In those correlation decay based FPTASes, one first constructs a tree structure and hopes to compute the marginal probability of the root. With a recursive relation, one writes the marginal probability of the root as a function of that of its sub-trees, then truncates the computation tree at certain depth and applies a rough guess at the leaf nodes. The correlation decay property ensures that the error for the root is exponentially small with respect to the depth of the tree, although there might be constant error for the leaves. To establish Theorem 2, we use a similar idea to construct a tree gadget so that the marginal probability (effective external field) for the root is very close to the target value. Using a tree recursion, one translates the target marginal probability for the root to that of its sub-trees. In the leaf nodes, we simply use some basic gadgets to approximate the target external field. Again, although these approximations for leaves may have constant gaps, the error at the root is exponentially small thanks to the correlation decay property. We believe that this idea of using an algorithm design technique to construct gadgets to establish hardness result is of independent interest and may find applications in other problems.

We also make some improvements on the algorithm side showing that there is an FPRAS if $\mu \le \gamma/\beta$. We remark that all the computational problem $\textsc{Ferro}(\beta,\gamma,\mu)$ and $\textsc{Ferro}(\beta,\gamma,\mathcal{V})$ is no more difficult than #BIS, as we can use the standard transformation to transform any ferromagnetic two-spin system to ferromagnetic Ising model with possibly different external fields and use the #BIS-easiness result in [7]. Thus, the two #BIS-hardness theorems can also be stated as #BIS-equivalent. We believe that the conjecture here is that for any fixed $\beta < \gamma$, there exists a critical μ_c such that it admits an FPRAS if the external field $\mu < \mu_c$, and it is #BIS-equivalent if $\mu > \mu_c$. The result of this paper is an important step towards this dichotomy.

Related Works and Organization of the Paper

The approximation for partition function of spin system and other similar models has been studied extensively [1, 19, 8, 11]. For the anti-ferromagnetic two-state spin model, the problem is known to be tractable up to the uniqueness threshold [16, 13, 18, 6], this includes the hard-core model as a special case [20, 17]. For the ferromagnetic two-state spin model, FPRAS was known for Ising model with arbitrary external field [10] and this was later extended to the whole ferromagnetic regime [9]. Besides the FPRASes, there is also a recent deterministic FPTAS for certain range of the parameters based on correlation decay and holographic reduction [15].

The remainder of the paper is organized as follows. We apply a reduction from a recent established hardness in [3] to prove Theorem 1 in Section 2. Based on this, we construct gadgets that can realize sufficiently small external field and prove Theorem 2 in Section 3. Finally, we present our improved tractable result in Section 4.

2 Bounded Local Fields

In this section, we show that spin systems with bounded local fields are already hard. The following theorem is a formal statement of Theorem 1.

▶ **Theorem 3.** *Let* $\beta < \gamma$, $\beta\gamma > 1$, $\Delta = \lfloor \frac{2\sqrt{\beta\gamma}}{\sqrt{\beta\gamma}-1} \rfloor + 1$ *and* $\mu > \left(\sqrt{\frac{\gamma}{\beta}}\right)^{\Delta}$. *Then* $\textsc{Ferro}\,(\beta,\gamma,[1,\mu])$ *is* #BIS*-hard.*[1]

We first introduce our starting point from anti-ferromagnetic Ising model on bipartite graphs in Section 2.1, and show the reduction in Section 2.2.

2.1 Anti-ferromagnetic Spin Systems on Bipartite Graphs

#BIS is a special anti-ferromagnetic two-state spin system. Similar to #BIS, one can also study other anti-ferromagnetic two-state spin systems on bipartite graphs. We use a prefix Bi- to emphasize that input graphs are bipartite, and a subscript Δ to indicate that maximum degree is Δ. For instance, the problem of $\textsc{Anti-Ferro}(\beta,\gamma,\mu)$ on bipartite graphs with maximum degree Δ is denoted shortly by $\textsc{Bi-Anti-Ferro}_\Delta(\beta,\gamma,\mu)$. The following theorem from [3] is the starting point of our reduction.

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[1] Technically, we should only define the problem by a finite set of external fields. In this paper and as in many others, we adopt the following convention: when we say a problem with an infinite set of external fields is hard, it means that there exists a finite subset of external fields to make the problem hard already.

▶ **Theorem 4** ([3]). *Suppose a set of anti-ferromagnetic parameters (β,γ,μ) lies in the non-uniqueness region of the infinite Δ-regular tree $\mathbb{T}_\Delta$. Then* Bi-Anti-Ferro$_\Delta(\beta,\gamma,\mu)$ *is* #BIS*-hard except for the case $(\beta=\gamma,\lambda=1)$.*

For simplicity, we use the special anti-ferromagnetic Ising model $\beta=\gamma<1$ in our reduction, for which the non-uniqueness condition is easy to state.

▶ **Proposition 5.** *If $\beta<\frac{\Delta-2}{\Delta}$, then there is a critical activity $\mu_c(\beta,\Delta)>1$ such that the Gibbs measure of Ising model (β,β,μ) on infinite Δ-regular tree $\mathbb{T}_\Delta$ is unique if and only if $|\log\mu|\geq\log\mu_c(\beta,\Delta)$.*

Proposition 5 is folklore, a proof can be found in, e. g. [16]. Combining the two results we get

▶ **Corollary 6.** *For all $0<\beta<1$, there is $\varepsilon>0$ such that for any $\mu\in(1,1+\varepsilon)$,* Bi-Anti-Ferro$_\Delta(\beta,\beta,\mu)$ *is* #BIS*-hard, where $\Delta=\lfloor\frac{2}{1-\beta}\rfloor+1$.*

Proof. As $\Delta=\lfloor\frac{2}{1-\beta}\rfloor+1$, we know that $\beta<\frac{\Delta-2}{\Delta}$. Then by Proposition 5, we can choose $\varepsilon=\mu_c(\beta,\Delta)-1$ to ensure that (β,β,μ) is in the non-uniqueness region of the infinite Δ-regular tree $\mathbb{T}_\Delta$ for all $\mu\in(1,1+\varepsilon)$. Then the corollary follows from Theorem 4. ◀

2.2 The Reduction

▶ **Lemma 7.** *For any $\beta<\gamma$ with $\beta\gamma>1$, $\mu>1$ and integer $\Delta>1$, we have*

$$\text{Bi-Anti-Ferro}_\Delta\left(\frac{1}{\sqrt{\beta\gamma}},\frac{1}{\sqrt{\beta\gamma}},\mu\right)\leq_{AP}\text{Bi-Ferro}_\Delta\left(\beta,\gamma,\left[\frac{1}{\mu}\sqrt{\frac{\gamma}{\beta}},\mu\left(\sqrt{\frac{\gamma}{\beta}}\right)^{\Delta}\right]\right).$$

Proof. Let bipartite graph $G(L\cup R,E)$ be an instance of Bi-Anti-Ferro$_\Delta\left(\frac{1}{\sqrt{\beta\gamma}},\frac{1}{\sqrt{\beta\gamma}},\mu\right)$. We construct an instance of ferromagnetic system with exactly the same graph. Each vertex $u\in L$ with degree d_u has weight $\mu\left(\sqrt{\frac{\gamma}{\beta}}\right)^{d_u}$, and each vertex $v\in R$ has weight $\frac{1}{\mu}\left(\sqrt{\frac{\gamma}{\beta}}\right)^{d_v}$. Then the maximum possible external field is $\mu\left(\sqrt{\frac{\gamma}{\beta}}\right)^{\Delta}$ while the minimum one is $\frac{1}{\mu}\sqrt{\frac{\gamma}{\beta}}$. Therefore, it is indeed an instance of Bi-Ferro$_\Delta\left(\beta,\gamma,\left[\frac{1}{\mu}\sqrt{\frac{\gamma}{\beta}},\mu\left(\sqrt{\frac{\gamma}{\beta}}\right)^{\Delta}\right]\right)$.

Let $Z_1(G)$ be the partition function of the anti-ferromagnetic Ising instance, and $Z_2(G)$ be that for the ferromagnetic system. We shall prove that $Z_1(G)=\gamma^{-|F|}\mu^{|R|}Z_2(G)$. Let $V\triangleq L\cup R$, $A=\begin{bmatrix}\frac{1}{\sqrt{\beta\gamma}} & 1\\ 1 & \frac{1}{\sqrt{\beta\gamma}}\end{bmatrix}$, $A'=\begin{bmatrix}\sqrt{\frac{\gamma}{\beta}} & \gamma\\ \gamma & \sqrt{\frac{\gamma}{\beta}}\end{bmatrix}$, $\hat{A}'=\begin{bmatrix}1 & \beta\\ \gamma & 1\end{bmatrix}$ and $\hat{A}=\begin{bmatrix}\beta & 1\\ 1 & \gamma\end{bmatrix}$. Then

$$\begin{aligned}
Z_2(G)&=\sum_{\sigma\in\{0,1\}^V}\prod_{(u,v)\in E}\hat{A}_{\sigma_u,\sigma_v}\prod_{u\in L}\left(\mu\left(\sqrt{\frac{\gamma}{\beta}}\right)^{d_u}\right)^{1-\sigma_u}\prod_{v\in R}\left(\frac{1}{\mu}\left(\sqrt{\frac{\gamma}{\beta}}\right)^{d_v}\right)^{1-\sigma_v}\\
&=\sum_{\sigma\in\{0,1\}^V}\prod_{(u,v)\in E}\hat{A}'_{\sigma_u,\sigma_v}\prod_{u\in L}\left(\mu\left(\sqrt{\frac{\gamma}{\beta}}\right)^{d_u}\right)^{1-\sigma_u}\prod_{v\in R}\left(\frac{1}{\mu}\left(\sqrt{\frac{\gamma}{\beta}}\right)^{d_v}\right)^{\sigma_v}\\
&=\sum_{\sigma\in\{0,1\}^V}\prod_{(u,v)\in E}A'_{\sigma_u,\sigma_v}\prod_{u\in L}\mu^{1-\sigma_u}\prod_{v\in R}\frac{1}{\mu^{\sigma_v}}\\
&=\mu^{-|R|}\gamma^{|F|}\sum_{\sigma\in\{0,1\}^V}\prod_{(u,v)\in E}A_{\sigma_u,\sigma_v}\prod_{u\in L}\mu^{1-\sigma_u}\prod_{v\in R}\mu^{1-\sigma_v}\\
&=\mu^{-|R|}\gamma^{|F|}Z_1(G).
\end{aligned}$$

Thus we can get an approximation for the anti-ferromagnetic Ising model by an oracle call to the ferromagnetic two-spin system. This concludes the proof. ◀

Now, for the target $\mu > \left(\sqrt{\frac{\gamma}{\beta}}\right)^{\Delta}$ in Theorem 3, we simply choose μ' close enough to 1 in Lemma 7 and Corollary 6, such that $\left[\frac{1}{\mu'}\sqrt{\frac{\gamma}{\beta}}, \mu'\left(\sqrt{\frac{\gamma}{\beta}}\right)^{\Delta}\right] \subseteq [1,\mu]$ and #BIS $\leq_{AP}$ Bi-Anti-Ferro$_\Delta\left(\frac{1}{\sqrt{\beta\gamma}}, \frac{1}{\sqrt{\beta\gamma}}, \mu'\right)$. Then we can conclude that Bi-Ferro$_\Delta(\beta,\gamma,[1,\mu])$ is #BIS-hard and complete the proof of Theorem 3.

3 Uniform Local Field

We establish Theorem 2 in this section. We distinguish between $\beta \leq 1$ and $\beta > 1$ cases, in Section 3.1 and 3.2 respectively.

3.1 The $\beta \leq 1$ case

We introduce a function $h(x) = \frac{\beta x+1}{x+\gamma}$ which is used throughout this section. Note that since $\beta\gamma > 1$, $h(x)$ is monotonically increasing and $\frac{1}{\gamma} < h(x) < \beta \leq 1$ for $x \in (0,+\infty)$. We shall prove the following key reduction.

▶ **Lemma 8.** *Let $\beta \leq 1, \beta\gamma > 1$, d be an integer such that $\beta(\beta\gamma)^d > 1$, μ^* be the largest solution of x to $x = \mu h(x)^d$, and $\mu > \frac{\gamma^d(\beta\gamma-1)}{\beta}\left(1+\frac{d+1}{\ln(\beta(\beta\gamma)^d)}\right)$. Then* Ferro $(\beta,\gamma,[1,\mu^*])$ $\leq_{AP}$ Ferro(β,γ,μ).

As $\mu^* = \mu h(\mu^*)^d$ and $\frac{1}{\gamma} < h(\mu^*) < \beta$, we have the following bound for μ^*.

▶ **Proposition 9.** $\frac{\mu}{\gamma^d} < \mu^* < \beta^d\mu$.

With this bound and Lemma 8, we can choose sufficiently large μ so that this μ^* is large enough to apply the hardness result (Theorem 3) of Ferro$(\beta,\gamma,[1,\mu^*])$ to get the hardness result for Ferro(β,γ,μ). Formally, we have

▶ **Theorem 10.** *Let $\beta \leq 1, \beta\gamma > 1$, d be an integer such that $\beta(\beta\gamma)^d > 1$, $\Delta = \lfloor\frac{2\sqrt{\beta\gamma}}{\sqrt{\beta\gamma}-1}\rfloor + 1$, and $\mu > \gamma^d \max\left\{\left(\sqrt{\frac{\gamma}{\beta}}\right)^{\Delta}, \frac{\beta\gamma-1}{\beta}\left(1+\frac{d+1}{\ln(\beta(\beta\gamma)^d)}\right)\right\}$. Then* Ferro$(\beta,\gamma,\mu)$ *is* #BIS*-hard.*

We remark that there always exists such integer d since $\beta > 0$ and $\beta\gamma > 1$. Different ds give different bounds for μ and it is not necessarily monotone. For a given β,γ, one can choose a suitable d to get the best bound[2].

In the remaining of this section, we prove the key reduction stated in Lemma 8. The main idea is to simulate any external field in $[1,\mu^*]$ by a vertex weight gadget. In Section 3.1.1 , we state the general framework of such simulation. Then in Section 3.1.2, we present the detailed construction of a gadget.

[2] We give one numerical example here to get some idea of this bound: if $\beta = 1$ and $\gamma = 2$, we can get $\Delta = 7$ and choose $d = 1$; then the theorem tells us that the problem Ferro$(1,2,\mu)$ is #BIS-hard if $\mu > 16\sqrt{2}$.

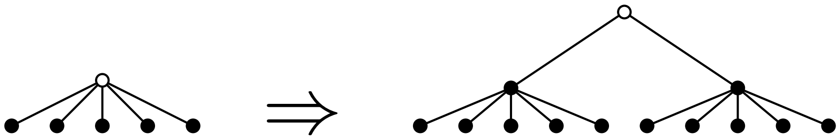

Figure 1 Result of $\mathcal{S}_5 \Rightarrow \texttt{comb}(\{\mathcal{S}_5, \mathcal{S}_5\})$, the output vertex is marked as unfilled.

3.1.1 Vertex Weight Gadget

▶ **Definition 11** (Vertex weight gadget)**.** Let $G(V,E)$ be a graph with a special output vertex v^*, define $\mu(G) = \frac{Z_G(v^*=0)}{Z_G(v^*=1)}$ where $Z_G(v^*=0)$ (resp. $Z_G(v^*=1)$) is the partition function of $G(V,E)$ in (β,γ,μ)-system conditioned on $v^*=0$ (resp. $v^*=1$). We call G a vertex weight gadget that realizes $\mu(G)$.

We also use a family of graphs to approach a given external field. Let $\{G_i\}_{i\geq 1}$ be a family of vertex weight gadgets. We say $\{G_i\}$ realizes μ if $\lim_{i\to\infty}\mu(G_i) = \mu$.

Vertex weight gadgets can be used to simulate external fields. Formally, we have the following reductions.

▶ **Lemma 12.** *Let G be a vertex weight gadget of $(\beta,\gamma,\mathcal{V})$. Then* $\textsc{Spin}(\beta,\gamma,\mathcal{V}\cup\{\mu(G)\}) \leq_{AP} \textsc{Spin}(\beta,\gamma,\mathcal{V})$.

Let $\{G_i\}$ be a sequence of vertex weight gadget of $(\beta,\gamma,\mathcal{V})$ to realize μ such that for any $\varepsilon > 0$ there is a G_i of size poly $\left(\varepsilon^{-1}\right)$ with $\exp(-\varepsilon) \leq \frac{\mu(G_i)}{\mu} \leq \exp(\varepsilon)$. *Then* $\textsc{Spin}(\beta,\gamma,\mathcal{V}\cup\{\mu\}) \leq_{AP} \textsc{Spin}(\beta,\gamma,\mathcal{V})$.

Proof. The proof of the first part is straightforward. For any instance H of $\textsc{Spin}(\beta,\gamma,\mathcal{V}\cup\{\mu(G)\})$ and a vertex of H with external field $\mu(G)$, we use one copy of G and identify the output vertex of G with that chosen vertex of H. After the identification, the external field in that vertex is that of output vertex of G. Therefore, after the modification, the new instance is an instance of $\textsc{Spin}(\beta,\gamma,\mathcal{V})$ and the partition function is equal to the partition function of H scaled by a polynomial-time computable global factor $\left(\frac{Z(G)}{1+\mu(G)}\right)^j$, where j is the number of vertices with external field $\mu(G)$ in H.

For the second part, for an instance H of $\textsc{Spin}(\beta,\gamma,\mathcal{V}\cup\{\mu\})$ and required approximation parameter ε, choose a gadget G_i which is $\varepsilon' = \frac{\varepsilon}{2n}$ close to realize μ; do the same modification as above using this G_i and call the oracle for the new instance with approximation parameter ε'. This gives the desired approximation for the original instance. ◀

3.1.2 The Construction

We first define a gadget operation `comb` as follows: for a given list of graphs $\mathcal{G} = \{G_1,\ldots,G_k\}$, each with output v_i^* for $i\in[k]$, $\texttt{comb}(\mathcal{G})$ is a new graph $G(V,E)$ that combines the graphs and joins their outputs. Fig. 1 is an illustration of `comb`. Formally, we define $V = \{u\}\cup\bigcup_{i\in[k]}V(G_i)$ and $E = \{(u,v_i^*)\mid i\in[k]\}\cup\bigcup_{i\in[k]}E(G_i)$, where u is the output of G. It is easy to verify that $\mu(G) = \mu\prod_{i\in[k]}h\left(\mu(G_i)\right)$.

We also define two basic gadgets. Let $\mathcal{S}_w$ be a w-star graph, with output being its center. In particular, $\mathcal{S}_0$ is the singleton graph. Note that $\mu(\mathcal{S}_w) = \mu h(\mu)^w$. We also define $\mathcal{T}_t$ be a d-ary tree with depth t. For any external field $\hat{\mu}\in(0,\mu^*]$, we shall construct a list of gadgets to simulate it. The two boundaries are approached by $\mathcal{S}_w$ and $\mathcal{T}_t$ respectively.

▶ **Proposition 13.** *Let $\mathcal{S}_w$ be a w-star and $\mathcal{T}_t$ be a d-ary tree with depth t. Then*

1. $\{\mathcal{S}_w\}_{w\geq 1}$ *realizes* 0*, or formally,* $\mu(\mathcal{S}_w) = \mu h(\mu)^w < \mu\beta^w$.
2. $\{\mathcal{T}_t\}_{t\geq 0}$ *realizes* μ^**, or formally, there exist two positive constants* ι *and* $c < 1$ *depending on* μ, β, γ *and* d *such that* $1 < \frac{\mu(\mathcal{T}_t)}{\mu^*} \leq \exp(c^t\iota)$.

Proof. (1) is obvious, we only prove (2).

Note that $\mu(\mathcal{T}_t) = \mu h(\mu(\mathcal{T}_{t-1}))^d$, we denote $f(x) = \mu h(x)^d$. Recall that μ^* is the largest fixed point of $f(x)$ and $f(\mu) < \mu$, we have $0 < f'(\mu^*) < 1$. Define $g(x) = \frac{xf'(x)}{f(x)}$, then $g(\mu^*) = f'(\mu^*)$. Since $g(x)$ is a continuous function, we can choose some $\eta > 0$ such that $0 < g(x) \leq c < 1$ for all $x \in (\mu^* - \eta, \mu^* + \eta)$.

We now define a sequence $\{x_i\}_{i\geq 0}$ such that $x_0 = \mu$ and $x_i = f(x_{i-1})$ for all $i \geq 1$. We claim that $\{x_i\}$ converges to μ^* as i approaches infinity. To see this, note that $x_{i+1} = f(x_i) < x_i$ and $x_i > \mu^*$ for all $i \geq 0$. This implies $\{x_i\}$ converges to some $z \geq \mu^*$. Moreover, since f is continuous, the sequence $\{f(x_i)\}_{i\geq 0}$ also converges to z. These two facts together imply $z = \lim_{i\to\infty} f(x_i) = f(\lim_{i\to\infty} x_i) = f(z)$. In other word, z is a fixed point of f and thus $z = \mu^*$. The claim implies that for some integer t_0, $x_{t_0} \in (\mu^*, \mu^* + \eta)$.

We define another sequence $\{y_i\}_{i\geq 0}$ such that $y_0 = \mu(\mathcal{T}_{t_0})$ and $y_i = f(y_{i-1})$ for all $i \geq 1$. It holds that $y_i \in (\mu^*, \mu^* + \eta)$ and thus $g(y_i) \leq c < 1$ for all $i \geq 0$. Therefore for all $t \geq 1$,

$$\begin{aligned}
\ln y_t - \ln \mu^* &= \ln f(y_{t-1}) - \ln f(\mu^*) \\
&= \frac{\tilde{y} f'(\tilde{y})}{f(\tilde{y})} \cdot |\ln y_{t-1} - \ln \mu^*| \quad \text{for some } y \in [\mu^*, y_{t-1}] \\
&= g(\tilde{y}) \cdot |\ln y_{t-1} - \ln \mu^*| \\
&\leq c \cdot |\ln y_{t-1} - \ln \mu^*| \\
&\leq c^t \eta.
\end{aligned}$$

We denote $\iota = \max\{\ln \mu, \eta c^{-t_0}\}$ and conclude the proof. ◀

Let d be the one that satisfies the requirement in the statement of Lemma 8. Our main idea to realize a target external field $\hat{\mu}$ is to construct a list of gadgets $\mathcal{G} = \{G_1, \ldots, G_d\}$ such that $\mu(\texttt{comb}(\mathcal{G})) \approx \hat{\mu}$ or more concretely $\hat{\mu} \approx \mu \prod_{i\in[d]} h(\mu(G_i))$. All but one of these G_i are basic gadgets of the following three types:

1. isolate point $\mathcal{S}_0$ with $\mu(\mathcal{S}_0) = \mu$;
2. $\mathcal{S}_w$ with large enough w such that $\mu(\mathcal{S}_w) \approx 0$; and
3. $\mathcal{T}_t$ with large enough t such that $\mu(\mathcal{T}_t) \approx \mu^*$.

The remaining one G_i is recursively constructed with a new target $\hat{\mu}'$ so that ideally $\hat{\mu} = \mu \prod_{i\in[d]} h(\mu(G_i))$ holds. The combination of these basic gadgets are carefully chosen so that the new target $\hat{\mu}'$ is also in the range $(0, \mu^*]$. Then we recursively simulate this $\hat{\mu}'$ by a subtree. We terminate the recursion after enough steps, and use a basic star gadget which is closest to the desired value as an approximation in the leaf. With a correlation decay argument, we show that the error in the root can be exponentially small in terms of the depth, although there may be a constant error in the leaf. A detailed construction with special treatment for the boundary cases are formally given in Algorithm 1. In the description of the algorithm and the analysis below, we denote $\alpha \triangleq \frac{\sqrt{\beta\gamma}-1}{\sqrt{\beta\gamma}+1} < 1$.

Before we prove that the construction is correct, we obtain a few observations which will be used in the proof. The condition on μ in Lemma 8 is due to the following property we need.

Algorithm 1: Constructing G_ℓ

```
function construct(ℓ, μ̂) :
input  : Recursion depth ℓ; Target 0 < μ̂ ≤ μ* to simulate;
output : Graph G_ℓ constructed.
begin
    if ℓ = 0 then
        Let k be the positive integer such that μh(μ)^{k+1} < μ̂ ≤ μh(μ)^k;
        return S_k;
    else
        Let k be the non-negative integer with μ*h(μ)^{k+1} < μ̂ ≤ μ*h(μ)^k ;
        Y' ← k · S_0; // a set of k copies of S_0.
        μ_1 ← μ̂ / h(μ)^k ;
        // Invariant: μh(x')^{d−i+1} = μ_i has a solution 0 < x' ≤ μ*.
        for i ← 1 to d − 1 do
            if μh(μ*)h(0)^{d−i} ≥ μ_i then
                y_i ← 0; w ← ⌊(ℓ·ln α − ln(dμ)) / ln β⌋ + 1; Y_i ← S_w;
            else
                y_i ← μ*; t ← ⌊(ℓ·ln α − ln d − ln ι) / ln c⌋ + 1; Y_i ← T_t;
            μ_{i+1} ← μ_i / h(y_i);
        Let μ̂' be the solution of μh(x) = μ_d in (0, μ*];
        Y ← Y' ∪ {Y_i}_{i≥1}^{d−1};
        δ ← exp(−(ln γ ln α / ln β) ℓ + (ln γ ln(dμ)) / ln β + ln(μ/γ));
        if μ̂' ≤ δ then
            Choose the largest integer w such that μ(1/γ)^w > δ;
            return comb(Y ∪ {S_w});
        else
            return comb(Y ∪ construct(ℓ − 1, μ̂'));
```

▶ **Proposition 14.** *Let* $\mu > \frac{\gamma^d}{\beta}(\beta\gamma - 1)\left(1 + \frac{d+1}{\ln(\beta(\beta\gamma)^d)}\right)$, *for any* μ_1 *with* $\mu^* h(\mu) < \mu_1 \le \mu^*$, *the equation* $\mu h(x)^d = \mu_1$ *always has a solution with* $0 < x \le \mu^*$.

Proof. It suffices to show $\mu \cdot h(0)^d \le \mu^* h(\mu)$ and $\mu \cdot h(\mu^*)^d \ge \mu^*$. Since $\mu^* = \mu h(\mu^*)^d$, the second part is trivial. As for the first part, it is sufficient to show $\left(\frac{h(\mu^*)}{h(0)}\right)^d h(\mu) > 1$. Note that $\left(\frac{h(\mu^*)}{h(0)}\right)^d h(\mu) > \gamma^d h(\mu^*)^{d+1} > \gamma^d \left(\beta - \frac{\beta\gamma-1}{\mu^*}\right)^{d+1}$,

$$\gamma^d \left(\beta - \frac{\beta\gamma - 1}{\mu^*}\right)^{d+1} > 1 \iff \ln\left(\beta(\beta\gamma)^d\right) + (d+1)\ln\left(1 - \frac{\beta\gamma - 1}{\beta\mu^*}\right) > 0,$$

$$(d+1)\ln\left(1 - \frac{\beta\gamma - 1}{\beta\mu^*}\right) \overset{(\clubsuit)}{>} -(d+1)\frac{\frac{\beta\gamma-1}{\beta\mu^*}}{1 - \frac{\beta\gamma-1}{\beta\mu^*}} \overset{(\spadesuit)}{>} -\ln\left(\beta(\beta\gamma)^d\right),$$

where ($\clubsuit$) is due to $\ln(1-x) > -\frac{x}{1-x}$ for $x \in (0,1)$, and ($\spadesuit$) is by the fact that $\beta(\beta\gamma)^d > 1$ and the choice of μ such that $-\frac{\beta\gamma-1}{\beta\mu^*} > \frac{\ln(\beta(\beta\gamma)^d)+d+1}{\beta\ln(\beta(\beta\gamma)^d)}$. ◀

▶ **Proposition 15.** *For every $x, t \geq 0$, it holds that $h(x+t) \leq (1+t)h(x)$ and $h((1+t)x) \leq (1+t)h(x)$.*

Proof. Note that $x, t \geq 0$,

$$h(x+t) \leq (1+t)h(x) \iff \left(\frac{\beta(x+t)+1}{x+t+\gamma}\right) \leq (1+t)\left(\frac{\beta x+1}{x+\gamma}\right)$$
$$\iff t^2(1+\beta x) + t\left(1+\gamma\left(1+\beta(x-1)\right)+x+\beta x^2\right) \geq 0.$$

Since $\left(1+\gamma\left(1+\beta(x-1)\right)+x+\beta x^2\right) > 0$, the inequality always holds.

$$h((1+t)x) \leq (1+t)h(x) \iff \frac{x(1+t)\beta+1}{x(1+t)+\gamma} \leq (1+t)\frac{\beta x+1}{x+\gamma}$$
$$\iff t^2(x+\beta x^2) + t(\gamma+2x+\beta x^2) \geq 0$$

Again every term is non-negative, the last inequality is always true. ◀

We first verify that the algorithm is well defined, namely $\mu h(x) = \mu_d$ does have a solution $\hat{\mu}'$ in $(0, \mu^*]$. This can be done by verifying the loop invariant "$\mu h(x')^{d-i+1} = \mu_i$ has a solution $0 < x' \leq \mu^*$" inductively.

Initialization. For $i = 1$, by Proposition 14, for some $0 < \tilde{x} \leq \mu^*$ it holds that $\mu h(\tilde{x})^{d-i+1} = \mu_i$.

Maintenance. Assuming $\mu h(\tilde{x})^{d-i+1} = \mu_i$ has solutions $\tilde{x} \in (0, \mu^*]$, we verify that $\mu h(x')^{d-i} = \mu_{i+1} \equiv \frac{\mu_i}{h(y_i)}$ has solutions $x' \in (0, \mu^*]$ for $i \in [1, d-1]$.

Case $\mu h(\mu^*)h(0)^{d-i} \geq \mu_i$. By assumption we have $\mu h(0)^{d-i+1} < \mu_i$, also note that $\mu_i \leq \mu h(\mu^*)h(0)^{d-i} \leq \mu h(0)h(\mu^*)^{d-i}$, hence $\mu h(0)^{d-i} < \frac{\mu_i}{h(0)} \leq \mu h(\mu^*)^{d-i}$. Then by continuity, $\mu h(x')^{d-i} = \frac{\mu_i}{h(0)}$ has solutions $0 < x' \leq \mu^*$.

Case $\mu h(\mu^*)h(0)^{d-i} < \mu_i$. By assumption $\mu h(\mu^*)^{d-i+1} \geq \mu_i$, thus $\mu h(0)^{d-i} < \frac{\mu_i}{h(\mu^*)} \leq \mu h(\mu^*)^{d-i}$, hence $\mu h(x')^{d-i} = \frac{\mu_i}{h(\mu^*)}$ has solutions $0 < x' \leq \mu^*$.

Termination. After the loop completes, $\mu h(x') = \mu_d$ has solutions $0 < x' \leq \mu^*$.

Now we verify the vertex weight gadget returned by the construction satisfies our requirement by choosing $\ell = O(-\log \varepsilon)$.

▶ **Lemma 16.** *For $0 < \hat{\mu} \leq \mu^*(\beta, \gamma, \mu)$, and let $G(V, E)$ be the graph returned by* `construct`$(\ell, \hat{\mu})$, *we have the following:*

1. $\exp(-(c+\ell)\cdot\alpha^\ell) \leq \frac{\mu(G)}{\hat{\mu}} \leq \exp((c+\ell)\cdot\alpha^\ell)$, *where $c = \ln\gamma$ and $\alpha = \frac{\sqrt{\beta\gamma}-1}{\sqrt{\beta\gamma}+1} < 1$;*
2. $|G| = \exp(O(\ell))$.

Proof. We apply induction on ℓ for both statements. We prove for (1) first. For the base case $\ell = 0$, we have

$$|\ln\mu(G) - \ln\hat{\mu}| \leq \left|\ln\mu h(\mu)^k - \ln\mu h(\mu)^{k+1}\right| = -\ln h(\mu) \leq \ln\gamma.$$

Assume that the statement holds for smaller ℓ. Let k, $\{y_i\}_{1\leq i\leq d-1}$ and $\{Y_i\}_{1\leq i\leq d-1}$ be parameters chosen in the algorithm. Define

$$F(z) = \ln\left(\mu h(\mu)^k \prod_{i=1}^{d-1} h(y_i)h(\exp(z))\right), \tilde{F}(z) = \ln\left(\mu h(\mu)^k \prod_{i=1}^{d-1} h(\mu(Y_i))h(\exp(z))\right).$$

We note that $F(z)$ is the *correct* recursion to compute $\ln(\mu(G))$ and $\tilde{F}(z)$ is our *approximate* recursion used in the algorithm.

In the following, we distinguish between $\hat{\mu}' \leq \delta$ and $\hat{\mu}' > \delta$.

- If $\hat{\mu}' \le \delta$, then $\ln \mu(G) = \tilde{F}(\ln \mu(\mathcal{S}_w))$ and $\ln \hat{\mu} = F(\ln \hat{\mu}')$. We have

$$\begin{aligned} F(\ln \hat{\mu}') \le \tilde{F}(\ln \mu(\mathcal{S}_w)) &= \ln \left(\mu h(\mu)^k \prod_{i=1}^{d-1} h(\mu(Y_i)) h(\mu(\mathcal{S}_w)) \right) \\ &\overset{(\heartsuit)}{\le} \alpha^\ell + \ln \left(\mu h(\mu)^k \prod_{i=1}^{d-1} h(y_i) h(\hat{\mu}') \right) \\ &= \alpha^\ell + F(\ln \hat{\mu}')), \end{aligned}$$

where $(\heartsuit)$ follows from the following facts derived from Proposition 15:

(i) If $y_i = 0$, then $0 \le \mu(Y_i) \le \frac{\alpha^\ell}{d}$, which implies $h(\mu(Y_i)) \ge h(y_i)$ and $h(\mu(Y_i)) \le \left(1 + \frac{\alpha^\ell}{d}\right) h(y_i) \le \exp\left(\frac{\alpha^\ell}{d}\right) h(y_i)$.

(ii) If $y_i = \mu^*$, then $\mu^* \le \mu(Y_i) \le \exp\left(\frac{\alpha^\ell}{d}\right) \mu^*$, which implies $h(\mu(Y_i)) \ge h(y_i)$ and $h(\mu(Y_i)) \le \exp\left(\frac{\alpha^\ell}{d}\right) h(y_i)$.

(iii) We claim that $\hat{\mu}' < \mu \left(\frac{1}{\gamma}\right)^w \le \mu(\mathcal{S}_w) \le \mu\beta^w \le \frac{\alpha^\ell}{d}$. The only nontrivial part is to verify that $\mu\beta^w \le \frac{\alpha^\ell}{d}$. Since w is the largest integer that $\hat{\mu}' < \mu \left(\frac{1}{\gamma}\right)^w$, we have $\mu \left(\frac{1}{\gamma}\right)^{w+1} \le \hat{\mu}'$, which gives $w \ge \frac{\ln \mu - \ln \delta}{\ln \gamma} - 1$. Plug this into $\mu\beta^w \le \frac{\alpha^\ell}{d}$ and let $\delta = \exp(-\frac{\ln \gamma \ln \alpha}{\ln \beta}\ell + \frac{\ln \gamma \ln(d\mu)}{\ln \beta} + \ln \frac{\mu}{\gamma})$, the inequality holds. Thus $h(\mu(\mathcal{S}_w)) \ge h(\hat{\mu}')$ and $h(\mu(\mathcal{S}_w)) \le h(\frac{\alpha^\ell}{d}) \le \left(1 + \frac{\alpha^\ell}{d}\right) h(\hat{\mu}') \le \exp\left(\frac{\alpha^\ell}{d}\right) h(\hat{\mu}')$.

- If $\hat{\mu}' > \delta$, define $x = \mu(\texttt{construct}(\ell - 1, \hat{\mu}'))$, then by induction hypothesis, it holds that $|\ln x - \ln \hat{\mu}'| \le (c + (\ell - 1)) \cdot \alpha^{\ell - 1}$.
Then similarly by Proposition 15 and the choice of w and t, we have $F(\ln x) \le \tilde{F}(\ln x) \le F(\ln x) + \alpha^\ell$. Thus by construction, we have

$$\begin{aligned} |\ln \mu(G) - \ln \hat{\mu}| &= \left|\tilde{F}(\ln x) - F(\ln \hat{\mu}')\right| \\ &\le \alpha^\ell + |F(\ln x) - F(\ln \hat{\mu}')| \\ &\le \alpha^\ell + |F'(\ln \tilde{x})| \cdot |\ln x - \ln \hat{\mu}'| \quad \text{(for some } \tilde{x} \in [\hat{\mu}', x].) \\ &\le \alpha^\ell + (\ell - 1) |F'(\ln \tilde{x})| \alpha^{\ell-1} + c |F'(\ln \tilde{x})| \alpha^{\ell-1} \end{aligned}$$

Thus it is sufficient to show that $|F'(\ln \tilde{x})| \le \alpha$. In fact, $F'(\ln x) = \frac{x \cdot h'(x)}{h(x)} = \frac{(\beta\gamma - 1)x}{(x+\gamma)(\beta x + 1)} \le \frac{\beta\gamma - 1}{\left(\sqrt{\beta\gamma} + 1\right)^2} = \alpha$.

Now we prove (2) of the Lemma. We denote $s(\ell) = \max_{\hat{\mu}} |\texttt{construct}(\ell, \hat{\mu})|$ and show that $s(\ell) = \ell \exp(O(\ell)) = \exp(O(\ell))$.

If $\ell = 0$, since $\hat{\mu}$ is either the eventual external field (which is a constant bounded away from 0), or $\hat{\mu} > \delta$, we have $s(\ell) = |\mathcal{S}_k| = O(1)$.

If $\ell > 0$, then $|Y_i| = \exp(O(\ell))$ and thus $|\mathcal{Y}| = \exp(O(\ell))$. By our choice of δ, it holds that $w = O(\ell)$ and thus $|S_w| = O(\ell)$. Therefore,

$$s(\ell) = \exp(O(\ell)) + \max\{s(\ell - 1), O(\ell)\} = \ell \exp(O(\ell)) = \exp(O(\ell)).$$

This concludes the proof. ◀

3.2 The $\beta > 1$ case

The $\beta > 1$ case follows a similar argument as that in [7] and is known as a folklore. We include a formal proof here to be self-contained.

▶ **Theorem 17.** *Let $\gamma > \beta > 1$ and $\mu > \frac{\gamma-1}{\beta-1}$. Then* $\textsc{Ferro}(\beta,\gamma,\mu)$ *is* #BIS*-hard.*

We follow the same idea of simulating external field and make use of Theorem 3. In the case $\beta > 1$, it is easy to see that we can simulate all positive external fields.

▶ **Lemma 18.** *For every $\hat{\mu} > 0$, there is a family of vertex weight gadgets $\{G_m\}_{m\geq 1}$ that realizes $\hat{\mu}$. Moreover, G_m is constructible in time $m^{O(1)}$ and*

$$\exp(-\frac{1}{m}) \leq \frac{\mu(G_m)}{\hat{\mu}} \leq \exp(\frac{1}{m}). \quad (1)$$

Proof. For any $m \geq 1$, we add x self-loops and y bristles to a single vertex v, where x and y are integers to be determined. Let v be the output of G_m, then $\mu(G_m) = \mu\left(\frac{\beta}{\gamma}\right)^x\left(\frac{\mu\beta+1}{\mu+\gamma}\right)^y$. Denote $a = \ln\frac{\gamma}{\beta}$, $b = \ln\frac{\mu\beta+1}{\mu+\gamma}$ and $c = \frac{\ln\hat{\mu}}{\ln\mu}$, then (1) is equivalent to

$$|(y \cdot b - x \cdot a) - c| \leq \frac{1}{m}.$$

We can use a procedure similar to extended Euclidean algoroithm to find such integers x, y in time $O(\ln m)$, such that it also guarantees $x, y = m^{O(1)}$. ◀

4 Improved Tractable Result

In this section, we establish the following tractable result:

▶ **Theorem 19.** *Let $\beta < \gamma$, $\beta\gamma > 1$ and $\mu \leq \gamma/\beta$. Then there is an FPRAS for* $\textsc{Ferro}(\beta,\gamma,\mu)$*.*

The proof of this theorem follows by refining the proof in [9], where they establish the tractable result for $\mu \leq (\gamma/\beta)^{\delta/2}$ for δ being the *minimum* degree of vertices in the graph. Specifically, we first contract all vertices with degree one and modify the external fields of their neighboring vertices, this only scales the partition function by a constant. Next, just as in [9], we shall reduce a (β,γ,μ) instance to a ferromagnetic Ising instance and apply the following celebrated result, which is first introduced in [10] for uniform external fields and refined for non-uniform external fields in [9]:

▶ **Theorem 20** ([10] and [9])**.** *There is an FPRAS for Ising system $(a, a, \mathcal{V})$ provided that $a > 1$ and all external fields in $\mathcal{V}$ are at most one.*

Let $G(V, E)$ be an instance of (β,γ,μ) system, we repeatedly apply the following operations until no degree one vertices can be found:

1. Pick a vertex u of degree one. Denote its incident edge by $e = (u, v)$. Let μ_u and μ_v be external fields on u and v respectively.
2. Remove u and edge (u, v), update $\mu_v \leftarrow \mu_v h(\mu_u)$.

Let $G'(V', E')$ be the remaining graph. G' either has no vertices of degree one, or it only contains a single vertex. Moreover, for every $v \in V'$, the external fields μ'_v satisfies $\mu'_v \leq \mu$. This can be easily verified given that $\mu \leq \gamma/\beta$. Let $\mathcal{U} = \{\mu'_v \mid v \in V'\}$, consider G' as an instance of $(\beta,\gamma,\mathcal{U})$ system, clearly $Z_{(\beta,\gamma,\mu)}(G) = Z^* \cdot Z_{(\beta,\gamma,\mathcal{U})}(G')$ where Z^* is an easily polynomial-time computable factor.

Let $\mathcal{V} = \left\{ \mu'_v \left(\frac{\beta}{\gamma}\right)^{d_v/2} \middle| v \in V' \right\}$ where d_v is the degree of v in G'. Let $\hat{G}(\hat{V}, \hat{E})$ be a copy of G' with $\hat{\mu}_v = \mu'_v \left(\frac{\beta}{\gamma}\right)^{d_v/2}$ for every $v \in \hat{V}$. We are going to verify that $Z_{(\beta,\gamma,\mathcal{U})}(G') = \sqrt{\frac{\gamma}{\beta}}^{|E'|} \cdot Z_{(a,a,\mathcal{V})}(\hat{G})$ for $a = \sqrt{\beta\gamma}$.

Define $A = \begin{bmatrix} \beta & 1 \\ 1 & \gamma \end{bmatrix}$, $A' = \begin{bmatrix} \gamma & \sqrt{\gamma/\beta} \\ \sqrt{\gamma/\beta} & \gamma \end{bmatrix}$ and $\hat{A} = \begin{bmatrix} \sqrt{\beta\gamma} & 1 \\ 1 & \sqrt{\beta\gamma} \end{bmatrix}$. Then,

$$\begin{aligned}
Z_{(\beta,\gamma,\mathcal{U})}(G') &= \sum_{\sigma \in \{0,1\}^{V'}} \prod_{(u,v)\in E'} A_{\sigma_u,\sigma_v} \prod_{v \in V'} {\mu'}_v^{1-\sigma_v} \\
&= \sum_{\sigma \in \{0,1\}^{V'}} \prod_{(u,v)\in E'} A'_{\sigma_u,\sigma_v} \prod_{v \in V'} \left(\left(\sqrt{\frac{\beta}{\gamma}}\right)^{d_v} {\mu'}_v \right)^{1-\sigma_v} \\
&= \sqrt{\frac{\gamma}{\beta}}^{|E'|} \sum_{\sigma \in \{0,1\}^{V'}} \prod_{(u,v)\in E'} \hat{A}_{\sigma_u,\sigma_v} \prod_{v \in V'} \left(\left(\sqrt{\frac{\beta}{\gamma}}\right)^{d_v} {\mu'}_v \right)^{1-\sigma_v} \\
&= \sqrt{\frac{\gamma}{\beta}}^{|E'|} \sum_{\sigma \in \{0,1\}^{\hat{V}}} \prod_{(u,v)\in \hat{E}} \hat{A}_{\sigma_u,\sigma_v} \prod_{v \in V'} \hat{\mu}_v^{1-\sigma_v} \\
&= \sqrt{\frac{\gamma}{\beta}}^{|E'|} \cdot Z_{(a,a,\mathcal{V})}(\hat{G})
\end{aligned}$$

Finally, to apply Theorem 20, we only need $\hat{\mu}_v \leq 1$ for all $v \in \hat{V}$. Recall that $\hat{G}$ has $\delta \geq 2$, hence $\mu \leq \gamma/\beta$ implies $\hat{\mu}_v \leq 1$. This concludes the proof.

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