On the Approximation Ratio of k-Lookahead Auction

Xue Chen[1,⋆], Guangda Hu[2,⋆], Pinyan Lu[3], and Lei Wang[4]

[1] Department of Computer Science, University of Texas at Austin
xchen@cs.utexas.edu
[2] Department of Computer Science, Princeton University
guangdah@cs.princeton.edu
[3] Microsoft Research Asia
pinyanl@microsoft.com
[4] Georgia Institute of Technology
lwang@cc.gatech.edu

Abstract. We consider the problem of designing a profit-maximizing single-item auction, where the valuations of bidders are correlated. We revisit the k-lookahead auction introduced by Ronen [6] and recently further developed by Dobzinski, Fu and Kleinberg [2]. By a more delicate analysis, we show that the k-lookahead auction can guarantee at least $\frac{e^{1-1/k}}{e^{1-1/k}+1}$ of the optimal revenue, improving the previous best results of $\frac{2k-1}{3k-1}$ in [2]. The 2-lookahead auction is of particular interest since it can be derandomized [2, 5]. Therefore, our result implies a polynomial time deterministic truthful mechanism with a ratio of $\frac{\sqrt{e}}{\sqrt{e}+1} \approx 0.622$ for any single-item correlated-bids auction, improving the previous best ratio of 0.6. Interestingly, we can show that our analysis for 2-lookahead is tight. As a byproduct, a theoretical implication of our result is that the gap between the revenues of the optimal deterministically truthful and truthful-in-expectation mechanisms is at most a factor of $\frac{1+\sqrt{e}}{\sqrt{e}}$. This improves the previous best factor of $\frac{5}{3}$ in [2].

1 Introduction

Optimal auction design is an important subject that has been heavily studied in both economics and theoretical computer science. Among the accomplished research in this area, a solid part is focused on *single-item auction*, which serves as a basic that provides insight to other more complicated problems. In the seminal paper [4], Myerson gave a complete characterization of the optimal single-item auction in the setting where bidders' valuations are drawn from independent distributions. However, the design of optimal auction with correlated bidders was left open.

From the economics aspect, a natural attempt for solving this problem is to generalize Myerson's characterization. Unfortunately, most results obtained via this approach are for restricted special cases, see [3] for a survey. One exception is [1] by Cremer and McLean where they relax the *individually rational* constraint and obtain mechanisms

⋆ This work was done when the authors were undergraduate students at Tsinghua university and intern students at Microsoft Research Asia.

N. Chen, E. Elkind, and E. Koutsoupias (Eds.): WINE 2011, LNCS 7090, pp. 61–71, 2011.

that extract the full social welfare. On the other hand, from a computer science aspect, two research directions (see [2]) were suggested.

The first one is the introduction of approximation algorithms into optimal auction design. In other word, instead of providing a characterization of the optimal auction, which might not even exist, one would look for efficient algorithms that guarantee the approximate optimality.

Along this direction, two computational models were considered-the *explicit model* [5] and the *oracle model* [6]. In the explicit model, the running time of an algorithm has to be polynomial in the support size of the distribution. However, in the oracle model, the algorithm is only allowed to make polynomial in the number of bidders queries to an oracle that returns the conditional distribution of a set of bidders given the values of the remaining ones. Ronen [6] gave the first efficient mechanism in the oracle model called 1-lookahead that 2-approximates the optimal revenue. In [7], Ronen and Saberi further proved that no deterministic efficient *ascending auction* can do better than $\frac{3}{4}$. On the other hand, in the explicit model, Papadimitriou and Pierrakos [5] showed that although there is an optimal deterministic auction among optimal truthful-in-expectation auctions for two bidders and this auction can be computed efficiently, it is NP-hard to find the optimal deterministic one for more than three bidders. The understanding the approximability of the optimal auction remains as a major challenge.

The second direction suggested is to relax the solution concept to *truthfulness-in-expectation*. One advantage of such relaxation is that the optimal truthful-in-expectation auction can be described as a linear program [2, 5] whose size is polynomial in the support of the distribution, hence can be computed efficiently in the explicit model.

Based on this observation, Dobzinski et.al. [2] studied a class of truthful-in-expectation mechanisms called *k-lookahead*. To be precise, for any fixed constant k, the k-lookahead mechanism runs the linear program among the k bidders with the highest bids, conditioning on the remaining bidders. Since k is a constant, the linear program can be solved efficiently in the oracle model.

In [2], the authors showed that the k-lookahead mechanism has approximation ratio $\frac{2k-1}{3k-1}$. As usual in computer science, improving this approximation ratio would be an important issue in this direction. Furthermore, a question that is of theoretical interest itself is the task of evaluating the gap between truthful-in-expectation and deterministically truthful mechanisms. Obviously, one would expect truthful-in-expectation mechanisms to achieve more revenue than the deterministic ones. Dobzinski et.al. [2] showed the gap is existed by providing an example of truthful-in-expectation mechanism that cannot be implemented as an universally truthful mechanism. At the same time, Papadimitriou et.al. [5] and Dobzinski et.al. [2] showed that there is an elegant derandomization of the 2-lookahead mechanism. In [2], Dobzinski showed that the gap is at most a factor of 5/3 between truthful-in-expectation and the optimal deterministically truthful. Closing the gap further requires either better truthful-in-expectation mechanisms that can be derandomized, or simply tighter analysis of the 2-lookahead mechanism.

Our results. In this paper, we contribute to both research directions mentioned earlier by providing more delicate analysis of the k-lookahead mechanisms in the oracle model. We show that the approximation ratio of k-lookahead mechanism is at least $\frac{e^{1-1/k}}{1+e^{1-1/k}}$, which improves the ratio given in [2]. In particular, our result implies that 2-lookahead mechanism is at least $\frac{\sqrt{e}}{1+\sqrt{e}}$-approximate and interestingly, we prove that our analysis is tight by showing an example in which 2-lookahead mechanism obtains exactly $\frac{\sqrt{e}}{1+\sqrt{e}}$ fraction of the optimal revenue.

Our analysis is based on the clever idea from [2] of comparing the revenue obtained by k-lookahead mechanism to the *t-fixed-price* and *t-pivot* auctions. The novelty of our approach is that instead of picking only one *threshold* t, we consider a series of thresholds $t_1, \ldots, t_m$ and choose the best series. Apparently, our analysis will lead to better ratio but become more complicated. Therefore, new idea and technique will be introduced for our analysis.

2 Preliminary

In this section, we formally define our problem and provide some useful facts that will be needed in the future discussion.

In a single-item auction, a seller wishes to sell one item to a group of n self-interested bidders. Each bidder has a private valuation $v_i \in \mathbb{R}^+$. We assume that there is a publicly known distribution $\mathcal{D}$ on the valuation space of the bidders. In this paper, we make no assumption on the distribution. In particular, bidders' valuations could be correlated. Since we only consider truthful mechanisms in this paper, we will equalize the notions of *bid* and *valuation*.

An auction M is a mechanism that takes a bid vector $\boldsymbol{v}$ and then decides who wins the item and for what price. We use $(\mathbf{x}, \boldsymbol{p})$ to denote the allocation and payment where $x_i(\boldsymbol{v})$ is the probability that bidder i gets the item and $p_i(\boldsymbol{v})$ is her expected payment. Here, the goal of each bidder i is to maximize her own *utility* defined as $x_i v_i - p_i$.

A mechanism is *deterministically truthful* if reporting the true valuation is a dominant strategy for each agent and $x_i(v) \in \{0, 1\}$ for every bidder i and every bid vector $\boldsymbol{v}$, and we say that a randomized mechanism is *universally truthful* if the mechanism is a probability distribution over deterministically truthful mechanisms. At last, *truthful-in-expectation* is a weaker notion in which an agent maximizes her expected utility by being truthful. It is easy to see that every deterministically truthful mechanism is universally truthful and every universally truthful mechanism is truthful in expectation.

In this paper, we are interested in designing truthful-in-expectation mechanisms. From now on, without particular specification, we will simply say a mechanism is *truthful* if it is truthful-in-expectation and an *optimal auction* is referred to a truthful-in-expectation mechanism that maximizes the seller's expected profit $E_{\mathcal{D}}[M] = E_{\boldsymbol{v}\sim\mathcal{D}}(\sum_{i=1}^{n} p_i(\boldsymbol{v}))$ on input distribution $\mathcal{D}$.

An useful observation is that the optimal auction can be described as a linear program [2, 5] that its size is polynomial in the support size of distribution. Therefore we can obtain an optimal auction in polynomial time in the size of distribution, which implies that the optimal auction can be computed efficiently in the explicit model. But the linear program is not generally efficient in the oracle model unless the number of bidders is a constant. This motivates the study of k-lookahead mechanisms [6, 2]. Due to the lack of space, we omit this linear program.

In a k-lookahead mechanism, we find the k bidders with the highest values. We denote the set of these k bidders by K. Next we get the conditional distribution D_K on $v_i \geq \max\{v_j | j \notin K\}$ for $i \in K$ and v_j is fixed for all $j \notin K$. Then we reject the bidders not in K and use the mentioned linear program for distribution D_K to get the allocation vector $\mathbf{x}_K$ and payment vector $\boldsymbol{p}_K$.

In this paper, we will investigate the approximation ratio of the k-lookahead mechanism. Here, we say an auction M is a c-approximation mechanism if $\frac{E_{\mathcal{D}}[M]}{E_{\mathcal{D}}[OPT]} \geq c$ where OPT is the revenue-maximizing valid auction on distribution $\mathcal{D}$.

Finally, the following theorem provides a characterization of deterministic mechanisms for single item auctions, which will be useful in the analysis of 2-lookahead mechanism.

Theorem 1. *[4] A deterministic mechanism, with allocation and payment rule $\boldsymbol{q}, \boldsymbol{p}$ respectively, is truthful if and only if for each bidder i and each v_{-i}, the following conditions hold:*

1. Monotone Allocation*: $q_i(v_i, v_{-i}) \leq q_i(v'_i, v_{-i})$ for all $v_i \leq v'_i$;*
2. Threshold Payment*: There exists a threshold $t_i(v-i)$ such that $p_i(v_i, v_{-i}) = t_i(v_{-i}) \cdot q_i(v_i, v_{-i})$.*

3 The Approximation Ratio

In this section, we present our main result. From now on, we fix a constant k and let K be the agents with the highest k bids. Let D_K be the conditional distribution of bidders in K conditioned on the remaining bidders. We show that the approximation ratio of k-lookahead mechanism is at least $\frac{e^{1-1/k}}{1+e^{1-1/k}}$.

Our high-level idea is to partition the optimal revenue into different components. Then we design several auctions that only sell the item in K and each of them approximately realizes part of the components. The revenues of these auctions provide a lower bound on the revenue of k-lookahead since it is the optimal auction that only sells the item to bidders in K. Without lose of generality, we assume $K = \{1, \cdots, k\}$ and v_{k+1} is the highest valuation not in K.

In the following, we always assume that the optimal revenue is 1. Now we consider the expected revenue of the optimal auction. As we mentioned before, we first partition the optimal revenue into four parts.

Definition 1. *Fix the optimal auction, for any $t > 1$, we define $L(t), \tilde{L}(t), M(t), H(t)$ as follows:*

1. *$L(t)$:the expected revenue from bidders in $N \backslash K$ for instances where no bidder in K has value at least $t \cdot v_{k+1}$.*
2. *$\tilde{L}(t)$:the expected revenue from bidders in $N \backslash K$ for instances where there are some bidders in K whose valuations are at least $t \cdot v_{k+1}$.*
3. *$M(t)$:the expected revenue from bidders in K for instances where no bidder in K has value at least $t \cdot v_{k+1}$.*
4. *$H(t)$:the expected revenue from bidders in K for instances where there are some bidders in K whose valuations are at least $t \cdot v_{k+1}$.*

Let the expected revenue from K in the optimal auction be $\alpha(\alpha \leq 1)$. By our definition, $M(t) + H(t) = \alpha$ and $L(t) + \tilde{L}(t) = 1 - \alpha$ for all $t \geq 1$.

Lemma 1. *The expected revenue of k-lookahead auction is at least α.*

Proof. Consider the following auction: If the optimal auction sells the item to bidder i in K with probability x_i and p_i, we still sell the item to i with probability x_i and ask for a payment p_i. Otherwise no one gets the item. This mechanism might not be truthful because it is possible that some bidders in $N \backslash K$ raises her bid so that she becomes a bidder in K and has a chance to get the item. To make this mechanism truthful, we raise the expected payment of each bidder i by $\max\{0, (v_{k+1} - p_i(v_{k+1}, v_{-i})) \cdot \frac{x_i(v_{k+1}, v_{-i})}{x_i(v)}\}$. This is then a truthful mechanism with expected revenue at least α. Furthermore, one can see that the mechanism only sells the item to bidders in K, therefore, the expected revenue of k-lookahead auction is at least α.

The above lemma provides a lower bound on the revenue of k-lookahead auction related to the components of M and H in the optimal auction. To get more such bounds, we need the following auctions first introduced by Dobzinski, Fu and Kleinberg [2]. Suppose there is a threshold $t \geq 1$:

t-Fixed Price Auction: Select a bidder j uniform from K at random. If any bidders in $K \backslash \{j\}$ have valuations no less than $t \cdot v_{k+1}$ then he gets the item with payment $t \cdot v_{k+1}$. If there are several bidders satisfy this condition, break ties arbitrary. Otherwise, bidder j gets the item with payment v_{k+1}.

t-Pivot Auction: Select a bidder j uniform from K at random. If any bidders in $K \backslash \{j\}$ have valuations no less than $t \cdot v_{k+1}$, we choose the bidder i with the smallest index. We run the *k-lookahead auction* on the conditional distribution D'_k that fix the valuations of bidders not in K, and require $v'_l \geq v_{k+1}(l \in K)$ and $v'_i \geq t \cdot v_{k+1}$. Otherwise, we allocate the item to bidder j with a payment v_{k+1}.

It is easy to verify that t-Fixed Price Auction is truthful. To check that t-Pivot Auction is truthful, the only case we should be careful is that a bidder i raises her valuation and let the mechanism run the k-lookahead auction. However, bidder i must be the only bidder whose valuation is not less than $t \cdot v_{k+1}$ in this case. So k-lookahead auction runs under the conditional distribution that $v_i \geq t \cdot v_{k+1}$. As a result, her payment must be at least $t \cdot v_{k+1}$, which exceeds her actual valuation. Therefore, t-Pivot auction is truthful.

In the following, we will choose a series of s threshold values $t_1 < t_2 < \cdots < t_s$ (whose values will be determined later) and relate the revenue of each t_i-Fixed Price Auction and t_i-Pivot Auction to the four components of the optimal revenue defined earlier.

To be precise, assume that $t_0 = 1$ and we define $M_i = M(t_i) - M(t_{i-1}) \geq 0$ which is the revenue from K in the optimal auction when the highest valuation is in $[t_{i-1} \cdot v_{k+1}, t_i \cdot v_{k+1})$.

Lemma 2. *The expected revenue of t_i-Fixed Price Auction is at least* $P_i = L(t_i) + \sum_{j=1}^{i} \frac{M_j}{t_j} + (\frac{k-1}{k} t_i + \frac{1}{k})(\tilde{L}(t_i) + \sum_{j=i+1}^{s} \frac{M_j}{t_j})$.

Proof. We consider two cases. In the first case, there is no bidder in K whose valuation is greater or equal than $t_i \cdot v_{k+1}$. So the auction allocates the item to the selected bidder j with payment v_{k+1}. The corresponding expected revenue in the optimal auction is $L(t_i) + \sum_{j=1}^{i} M_j$. From the definition of M_i, the revenue of our auction is at least $L(t_i) + \sum_{j=1}^{i} \frac{M_j}{t_j}$.

In the second case, there are some bidders whose valuations are at least $t_i \cdot v_{k+1}$. In our auction, the auction will obtain $t_i \cdot v_{k+1}$ with probability at least $\frac{k-1}{k}$. Otherwise the auction will obtain at least v_{k+1}. Therefore the expected revenue of this auction is at least $(\frac{k-1}{k}t_i + \frac{1}{k})\tilde{L}(t_i)$ when the optimal auction allocates the item to bidders not in K. At the same time, the expected revenue of this auction is at least $(\frac{k-1}{k}t_i + \frac{1}{k})\sum_{j=i+1}^{s} \frac{M_j}{t_j}$ when the optimal auction allocates the item to K.

From all discussion above, the expected revenue of t_i-Fixed Price Auction is at least $P_i = L(t_i) + \sum_{j=1}^{i} \frac{M_j}{t_j} + (\frac{k-1}{k}t_i + \frac{1}{k})\tilde{L}(t_i) + (\frac{k-1}{k}t_i + \frac{1}{k})\sum_{j=i+1}^{s} \frac{M_j}{t_j}$.

Similarly, we can prove the following:

Lemma 3. *The expected revenue of t_i-Pivot Auction is at least* $Q_i = L(t_i) + \sum_{j=1}^{i} \frac{M_j}{t_j} + \frac{k-1}{k}H(t_i) + \frac{1}{k}(\tilde{L}(t_i) + \sum_{j=i+1}^{s} \frac{M_j}{t_j})$.

Let $R_i = \max\{P_i, Q_i\}$ and we can see that $\max_{1 \le i \le s} R_i$ is a lower bound on the revenue of k-lookahead. From the above lemma, this lower bound is explicitly related to the components M, H, L and $\tilde{L}$. In the following, we will choose s large enough and $t_1, \ldots, t_s$ appropriately to obtain a lower bound on $\max_{1 \le i \le s} R_i$ that is only related to α. Together with Lemma 1, we will get the desired approximation ratio. Now we prove this lower bound:

Lemma 4. $\max_{1 \le i \le s} R_i \ge 1 - e^{-(1-1/k)}\alpha$.

Proof. To prove this lemma, we need to eliminate the explicit dependency of $\max_{1 \le i \le s} R_i$ to the components of M, H, L and $\tilde{L}$.

First of all, for each t_i, we can replace $\tilde{L}(t_i), H(t_i)$ by $1 - \alpha - L(t_i), \alpha - M(t_i)$ and simplify P_i, Q_i as:

$$P_i = (t_i + \frac{1}{k})(1-\alpha) - (\frac{k-1}{k}t_i + \frac{1}{k} - 1)L(t_i) + \sum_{j=1}^{i} \frac{M_j}{t_j} + \sum_{j=i+1}^{s} (\frac{k-1}{k}t_i + \frac{1}{k})\frac{M_j}{t_j}$$

$$Q_i = \alpha + \frac{1}{k}(1-\alpha) + \frac{k-1}{k}L(t_i) + \sum_{j=1}^{i}(\frac{1}{t_j} - \frac{k-1}{k})M_j + \sum_{j=i+1}^{s} \frac{1}{k}\frac{M_j}{t_j}$$

Now we are ready to eliminate $L(t_i)$. Since $R_i = \max\{P_i, Q_i\}$, we have

$$R_i \ge \frac{1}{t_i}P_i + \frac{t_i - 1}{t_i}Q_i = 1 - (\frac{k-1}{kt_i} + \frac{1}{k})\alpha + \sum_{j=1}^{s} \frac{M_j}{t_j} - \frac{k-1}{k}(1 - \frac{1}{t_i})\sum_{j=1}^{i} M_j.$$

At last, we will eliminate M_j for all j. Observe that $\max_{1 \le i \le s} R_i$ is lower bounded by the average, we have the following:

$$\max_{1\le i\le s} R_i \ge \sum_{i=1}^{s} R_i \ge s - \sum_{i=1}^{s}(\frac{k-1}{kt_i}+\frac{1}{k})\alpha + \sum_{j=1}^{s}\big[\frac{s}{t_j} - \sum_{i=j}^{s}\frac{k-1}{k}(1-\frac{1}{t_i})\big]M_j$$

Therefore, in order to eliminate M_j for all j, we only need to choose numbers $t_1, \dots, t_s$ such that

$$\frac{s}{t_j} - \sum_{i=j}^{s}\frac{k-1}{k}(1-\frac{1}{t_i}) = 0, \quad \text{for all } 1 \le j \le s. \tag{1}$$

As long as we can find such $t_1, \dots, t_s$, we have:

$$\max_{1\le i\le s} R_i \ge 1 - \big[\frac{k-1}{k}\cdot\frac{1}{s}\sum_{i=1}^{s}\frac{1}{t_i} + \frac{1}{k}\big]\alpha. \tag{2}$$

At first, we use β to denote $\frac{k-1}{k}$ and have $\frac{s}{t_s} - \beta(1-\frac{1}{t_s}) = 0$ when $j = s$. So $1-\frac{1}{t_s} = \frac{s}{s+\beta}$. Then, comparing the equations of j and $j+1$, we obtain $(1-\frac{1}{t_j}) = (\frac{s}{s+\beta})^{s-j+1}$. Therefore $\sum_{i=1}^{s}\frac{1}{t_i} = s - \sum_{i=1}^{s}(\frac{s}{s+\beta})^i = s - \frac{s}{\beta}(1-(\frac{s}{s+\beta})^s)$. At the same time, we know $\lim_{s\to\infty}(\frac{s}{s+\beta})^s = e^{-\beta} = e^{-\frac{k-1}{k}}$

Together with (2), we have $\max_{1\le i\le s} R_i \ge 1 - e^{-(1-1/k)}\alpha$.

Finally, we are ready to prove:

Theorem 2. *The approximation ratio of k-lookahead mechanism is at least* $\frac{e^{1-1/k}}{1+e^{1-1/k}}$.

Proof. Let rev_k be the revenue of the k-lookahead mechanism. From Lemma 4, we know that $\mathsf{rev}_k \ge 1 - e^{-(1-1/k)}\alpha$. Together with Lemma 1, we have

$$\mathsf{rev}_k \ge \max\{\alpha, 1 - e^{-(1-1/k)}\alpha\}.$$

Simple calculation shows that for all positive value x, $\max\{\alpha, 1-x\alpha\} \ge \frac{1}{1+x}$. Therefore, we have $\mathsf{rev}_k \ge \frac{e^{1-1/k}}{1+e^{1-1/k}}$. This completes our proof.

4 Tightness of Analysis

In the previous section, we showed that the approximation ratio of k-lookahead is $\frac{e^{1-1/k}}{1+e^{1-1/k}}$. In particular, the 2-lookahead mechanism, which is of special interest, has an approximation ratio of at least $\frac{\sqrt{e}}{1+\sqrt{e}}$. In this section, we design an example to show that our analysis for 2-lookahead is tight.

First of all, we need some definitions. Because 2-lookahead auction can be made deterministically [2, 5], it either allocates the item, or does not allocate to anyone. So we only consider the 2-lookahead mechanisms that are deterministic from now on. We will consider the *empty instances* that a 2-lookahead mechanism doesn't allocate the item. We use $\mathsf{empt}(D)$ to denote the maximal *empty probability* over 2-lookahead mechanism that empty instances occur on a distribution D. In the following, we will use

$E_2(D)$ to denote the 2-lookahead mechanism with the maximum empty probability for distribution D.

In a setting where there are only three bidders, we say that a distribution D is *valid*, if the third bidder always has valuation $v_3 = 1$ and the valuations of the other two bidders are at least 1.

We first prove a property of valid distributions. Let $\mathsf{rev}_2(D)$ and $\mathsf{opt}(D)$ denote the revenue of the 2-lookahead and the optimal auction for a distribution D respectively.

Lemma 5. *Let D be a valid distribution on three bidders, then* $\mathsf{opt}(D) \geq \mathsf{rev}_2(D) + \mathsf{empt}(D)$.

Proof. Consider this auction $\mathcal{A}$: run the 2-lookahead auction $E_2(D)$ and if it allocates the item to bidder i in $K = \{1, 2\}$ with payment p, we still allocate the item to i with payment p. Otherwise we allocate the item to bidder 3 with payment 1. It is easy to see that $\mathcal{A}$ is truthful, and its revenue is $\mathsf{rev}_2(D) + \mathsf{empt}(D)$. Therefore, $\mathsf{opt}(D) \geq \mathsf{rev}_2(D) + \mathsf{empt}(D)$.

The above lemma provides a lower bound of opt. In the following, we will explicitly construct a valid distribution D such that $\mathsf{empt}(D) \geq \frac{\mathsf{rev}_2(D)}{\sqrt{e}}$, hence prove our desired ratio.

In our example, there are three bidders and the third bidder's valuation is always 1. Now we construct the distribution D_2 for the first two bidders explicitly. We assume that there are m possible valuations $p_0, p_1, \cdots, p_m$(m is an *odd number*). Then we define $x_0 = 1$ and $x_i = (1+p)^i - (1+p)^{i-1} = p(1+p)^{i-1}$ for $1 \leq i \leq m$ where p is a parameter. We will set the value of p and choose $p_1, \ldots, p_m$ later. One can see that a property of our construction is $\sum_{0 \leq i \leq j} x_i = (1+p)^j$ for all $j \leq m$.

Now we consider this following distribution D_2 where $D_2(i, j)$ denotes the probability of $v_1 = p_i, v_2 = p_j$:

$$D_2(i,j) = \begin{cases} x_i x_j & i+j < m \\ x_i(\sum_{j \leq k \leq m} x_k) & (i+j = m) \text{ and } (i < j) \\ (\sum_{i \leq k \leq m} x_k) x_j & (i+j = m) \text{ and } (i > j) \\ 0 & i+j > m \end{cases}$$

In fact, D_2 should be normalized to become a distribution. However, since we only care about the ratio between $\mathsf{empt}(D)$ and $\mathsf{rev}_2(D)$, we will simply use D_2 as the distribution without normalizing. From now on, we will simply use E_2, rev_2 and empt to denote $E_2(D)$, $\mathsf{rev}_2(D)$ and $\mathsf{empt}(D)$.

Now we choose $p_0 = 1$ and $p_i = \sum_{0 \leq j \leq m} x_j / \sum_{i \leq j \leq m} x_j$ for all $1 \leq i \leq m$. Therefore, we have $p_0 \leq p_1 \leq \cdots \leq p_m$. Furthermore,we obtain the following characterization of the event that E_2 allocates the item:

Lemma 6. *Let p_i, p_j be the bid of bidder 1 and 2 respectively, then E_2 allocates the item if and only if $i + j = m$.*

Proof. First of all, by our choice, it is easy to verify the following:

Property 1. If $i < m/2$, then: $p_0(\sum_{k=0}^{m-i} D_2(i,k)) = \cdots = p_j(\sum_{k=j}^{m-i} D_2(i,k)) = \cdots = p_{m-i} D_2(i, m-i) = x_i \sum_l x_l$.

Basically, this property can be interpreted as follows: fix $v_1 = p_i$, the expected revenue obtained by offering bidder 2 a threshold price p_j is a constant when $0 \leq j \leq m - i$. As a result, recall that by Theorem 1, the winner in a single item auction pays the threshold price, we have:

Corollary 1. *In E_2, $t_2(v_1) \geq p_{m-i}$ when $v_1 = p_i$ for all $i < m/2$. Similarly, $t_1(v_2) \geq p_{m-j}$ when $v_2 = p_j$ for all $j < m/2$.*

The proof of the corollary is straightforward: Suppose $v_1 = p_i$ for some $i < m/2$. If $t_2(v_1) < p_{m-i}$, then we can always increase the threshold price to p_{m-i} without decreasing the revenue. By doing this, we only increase the empty probability. This is a contradiction to our assumption that E_2 maximizes the empty probability.

Now we are ready to prove the lemma. If it is not true, suppose $i + j < m$ but E_2 allocates the item to either bidder 1 or 2. Consider the smallest sum of $i+j$ that satisfies the above. Without lose of generality, we may assume $i < m/2$. From Corollary 1, since $i + j < m$, we know that bidder 2 can not get the item. Therefore, bidder 1 gets the item when $v_1 = p_i$ and $v_2 = p_j$. At the same time, $j > m/2$ otherwise we can get a contradiction from Corollary 1. So bidder 1 still gets the item when $v_1 = p_{m-j} > p_i$ and $v_2 = p_j$.

Now we show that we can modify the allocation of E_2 when $v_2 = p_j$ to get more empty probability and the expected revenue of modified auction is not less than the original one. Let E_2' be an auction as follows: (1) it performs exactly the same as E_2 when $v_2 \neq p_j$ and (2) when $v_2 = p_j$, E_2' allocates the item to bidder 2 only when $v_1 = p_{m-j}$ and otherwise allocates nothing.

Obviously, E_2' has a larger empty probability than E_2. To get a contradiction, we only need to prove that its expected revenue is at least as large as E_2. In other words, we want to show:

$$p_j D_K(m - j, j) \geq p_i \sum_{k=i}^{m-j} D_K(k, j) \tag{3}$$

By our construction, simple calculation shows that (3) is equivalent to the following

$$p(1 + p)^{m-j-1}\big((1 + p)^{j-1} - \sum_{l=0}^{i-1} x_l\big) \geq p(1 + p)^{j-1}\big((1 + p)^{m-j-1} - \sum_{l=0}^{i-1} x_l\big),$$

which always holds when $j > m/2$. This is a contradiction.

By the above characterization, we can easily calculate rev_2 and empt. We will show that by choosing the parameter p appropriately, $\mathsf{rev}_2 \leq \sqrt{e} \cdot \mathsf{empt}$, which implies:

Theorem 3. *The approximation ratio of 2-lookahead auction is at most $\frac{\sqrt{e}}{\sqrt{e}+1}$.*

Proof. By Lemma 6 and our construction, we first estimate rev_2 as follows:

$$\mathsf{rev}_2 \leq \sum_{i,j:i+j=m} \max\{p_i, p_j\} D_2(i, j) = 2\Big(\sum_{0 \leq i < m/2} x_i\Big)\Big(\sum_{l=0}^{m} x_l\Big) = 2(1 + p)^{3m/2}.$$

Now we compute the empty probability empt. Again, by Lemma 6, we have empt $= \sum_{i,j:i+j<m} x_i x_j$, which can be calculate as follows:

$$\begin{aligned}\sum_{i,j:i+j<m} x_i x_j &= \sum_{j=0}^{m-1} x_j + \sum_{i=1}^{m-1}\sum_{j=0}^{m-1-i} x_i x_j \\ &= (1+p)^{m-1} + (m-1)p(1+p)^{m-2}.\end{aligned}$$

We set $p = 1/m$ and let $m \to \infty$, we have $\mathsf{rev}_2 \leq 2e^{3/2}$ and $\mathsf{empt} \geq 2e$. Therefore, $\mathsf{rev}_2 \leq \sqrt{e} \cdot \mathsf{empt}$. Therefore, by our previous argument, the approximation ratio of 2-lookahead auction is at most $\frac{\sqrt{e}}{\sqrt{e}+1}$.

5 Discussion and Open Questions

Perhaps the first question that every theoretical computer scientist would ask here is whether the analysis of the k-lookahead mechanism can be improved in general. An important open problem is whether the approximation ratio of k-lookahead mechanism tends to 1 when k tends to positive infinity. A nature attempt for this question from the negative aspect is to generalize our tight instance for 2-lookahead mechanism in section 4 to the k-lookahead mechanism for general k. In particular, one might consider the following distribution $D_K(i_1, ..., i_k)$ for the set K of the highest k bidders:

1.$D_K(i_1, \cdots, i_k) = 0$: there exists $p, q \in [k]$ such that $p \neq q$ and $i_p + i_q > m$.
2.$D_K(i_1, \cdots, i_k) = \prod_{j \in K} x_{i_j}$: for all $p, q \in [k](p \neq q)$, we have $i_p + i_q < m$.
3.$D_K(i_1, \cdots, i_k) = \prod_{j \in K \setminus \{l\}} x_{i_j} \cdot \sum_{j=i_l}^{m} x_j$: $\max_{p,q \in [k](p \neq q)} \{i_p + i_q\} = m$, where $i_l = \max\{i_1, \cdots, i_k\}$.

Again, we assume that the highest bid outside K is $v_{k+1} = 1$. Similar to the analysis for 2-lookahead, we can prove that k-lookahead allocates to some bidder in K if and only if $i_1, i_2, \cdots, i_k$ is such that $\max_{p,q \in [k](p \neq q)} \{i_p + i_q\} = m$. However, simple calculation implies that the ratio between the empty probability and the revenue of the k-lookahead is at most $2/k$ when k tends to infinity. This only implies a $\frac{k}{k+2}$ upper bound on the approximation ratio of the k-lookahead mechanism. Therefore, to obtain better upper bound, if possible, one might need new ideas and techniques.

From the positive aspect, one might improve the analysis via the following approach: Instead comparing the revenue of k-lookahead to t-fixed price and t-pivot auctions, we could compare to more delicate auctions such as a hybrid of t_1-fixed price and t_2-pivot auctions for distinct values of t_1, t_2.

Another interesting open question is to further close the gap between the revenues of the optimal deterministically truthful and truthful-in-expectation mechanisms. Our analysis of 2-lookahead implies that the gap is at most a factor of $\frac{1+\sqrt{e}}{\sqrt{e}}$. As we mentioned, our analysis is tight, hence closing the gap further requires better truthful-in-expectation mechanisms which can be derandomized.

References

1. Cremer, J., McLean, R.P.: Optimal selling strategies under uncertainty for a discriminating monopolist when demands are interdependent. Econometrica 53(2), 345–361 (1985)
2. Dobzinski, S., Fu, H., Kleinberg, R.D.: Optimal auctions with correlated bidders are easy. In: Proceedings of the 43rd Annual ACM Symposium on Theory of Computing, STOC 2011, pp. 129–138 (2011)
3. Klemperer, P.: Auction theory: A guide to the literature. Microeconomics, EconWPA (March 1999)
4. Myerson, R.B.: Optimal acution design. Mathematics of Operations Research 6(1), 58–73 (1981)
5. Papadimitriou, C.H., Pierrakos, G.: On optimal single-item auctions. In: Proceedings of the 43rd Annual ACM Symposium on Theory of Computing, STOC 2011, pp. 119–128 (2011)
6. Ronen, A.: On approximating optimal auctions. In: Proceedings of the 3rd ACM Conference on Electronic Commerce, EC 2001, pp. 11–17 (2001)
7. Ronen, A., Saberi, A.: On the hardness of optimal auctions. In: Proceedings of the 43rd Symposium on Foundations of Computer Science, FOCS 2002, pp. 396–405 (2002)