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# Envy-Free Pricing with General Supply Constraints for Unit Demand Consumers

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**Abstract** The envy-free pricing problem can be stated as finding a pricing and allocation scheme in which each consumer is allocated a set of items that maximize his/her utility under the pricing. The goal is to maximize seller revenue. We study the problem with general supply constraints which are given as an independence system defined over the items. The constraints, for example, can be a number of linear constraints or matroids. This captures the situation where items do not pre-exist, but are produced in reflection of consumer valuation of the items under the limit of resources. This paper focuses on the case of unit-demand consumers. In the setting, there are $n$ consumers and $m$ items; each item may be produced in multiple copies. Each consumer $i \in [n]$ has a valuation $v_{ij}$ on item $j$ in the set $S_i$ in which he/she is interested. He/she must be allocated (if any) an item which gives the maximum (non-negative) utility. Suppose we are given an $\alpha$-approximation *oracle* for finding the maximum weight independent set for the given independence system (or a slightly stronger oracle); for a large number of natural and interesting supply constraints, constant approximation algorithms are available. We obtain the following results. 1) $O(\alpha \log n)$-approximation for the general case. 2) $O(\alpha k)$-approximation when each consumer is interested in at most $k$ distinct types of items. 3) $O(\alpha f)$-approximation when each type of item is interesting to at most $f$ consumers. Note that the final two results were previously unknown even without the independence system constraint.

**Keywords** envy-free pricing, approximation, matroid

## 1 Introduction

Every company is an entity that has the goal of maximizing revenues, and faces two fundamental problems, namely producing and pricing items. The limitation of resources such as materials or human resources often restricts items that can be produced. For example, there may be a limit on the maximum number of items per group of items that can be delivered. Another possibility is that items may consume different types and amount of resources during production.

Another goal we seek to achieve, together with the maximization of revenue, when pricing items, is not to disappoint consumers by offering an insufficient supply. We assume that every consumer will buy certain items that maximize his/her utility, which is defined as his/her valuation of the items minus their prices. That is, pricing must guarantee that consumers are allocated the items that they most prefer under the pricing. When the supply of items is pre-given, such a pricing scheme is known as envy-free pricing[1].

This paper initiates a study of the problem of revenue maximization for producing items under general supply constraints and pricing them in an envy-free fashion. We assume that there are $n$ consumers and that each consumer's valuation of the items is known. There are $m$ distinct types of items that can be produced. We will use $[n]$ and $[m]$ to denote the set of consumers and the distinct types of items respectively. We model the supply constraints as an independence system. The downward closure property of the system is natural, i.e., if a set of items can be produced, so can its subsets. The independence system captures a variety of interesting constraints such as a number of linear constraints or matroids.

The envy-free pricing problem in the absence of the general supply constraints has been studied primarily for two special cases. This is because of computational issues and also because each consumer's valuation on all subset of items in general cannot be specified in a

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tractable size. The first case is the unit demand consumers case (UD) in which each consumer $i$ is allocated at most one item from among the items $S_i$ of interest. This arises when $S_i$ consists of similar items, so one item can fully meet consumer $i$'s need. The other case is the single-minded consumers case (SM), in which each consumer is interested in a bundle of items $B_i$. The bundle $B_i$, only as the entire bundle, has some value to consumer $i$; the partial acquisition of the bundle has no value to him/her. This captures the situation when the items in $B_i$ are complement to each other.

We will examine the unit demand consumers case when the supply is constrained by an independence system. We consider the general unit demand case together with two interesting special cases. The first case is where each consumer is interested in at most $k$ distinct types of items, i.e., $|S_i| \leqslant k$ for all $i$, which we will call the unit demand with bounded set size (UD-BSS). The other case we call the unit demand with bounded frequency (UD-BF), meaning that each item is of interest to a maximum of $f$ consumers, i.e., $|\{i \in [n] \mid j \in S_i\}| \leqslant f$ for all $j \in [m]$. In other words, this is the case when only a small number of users compete for each type of item.

*Our Results.* Assuming that we have an $\alpha$-approximation oracle for finding the maximum weight independent set $X$ in $\mathcal{I}$, we show how to deal with the envy-free pricing problem that is constrained on the independence system. We may require a slightly stronger version of the approximation oracle that has one more matroid constraint, but which still captures many interesting cases for which good approximations are available. As regards the general unit demand case, the bounded set size case and the bounded frequency case, we give $O(\alpha \log n)$, $O(\alpha k)$ and $O(\alpha f)$ approximations, respectively. In order to emphasize the general supply constraints under consideration, we add the suffix "-C" to each problem name.

Several results were given for pre-existing items, i.e., each item has an individual supply limit. Guruswami *et al.* gave an $O(\log n)$-approximation for UD by making a clever use of Walrasian equilibrium[1]. Briest[2] showed that for a certain constant $\epsilon > 0$, it is unlikely that there exists an approximation better than $O(\log^{\epsilon} n)$ under a certain complexity assumption. Table 1 summarizes our results with and without the supply constraint $\mathcal{I}$, together with the previously known results. We note that Briest's inapproximability result for UD-BF case is not formally stated in the paper, but is implied by the hardness instance.

Our algorithms and analysis borrow some ideas from [1, 3-5]. Similar to the case in [1], we make crucial use of the connection between Walrasian equilibria and the envy-free pricing for the unit demand case. The random sampling (partitioning) technique used in [4-5] will play a role in our algorithms and analysis. However, it is non-trivial to incorporate the general independence system; the results in [4-5] are for the unlimited-supplied single-minded consumers case.

**Table 1.** Summary of Our Results for the Unit Demand Case

| | UD-C | UD-BSS-C | UD-BF-C |
|---|---|---|---|
| Upper Bound | $O(\alpha \log n)^*$ | $O(\alpha k)^*$ | $O(\alpha f)^*$ |
| | UD | UD-BSS | UD-BF |
| Upper Bound | $O(\log(n))$[1] | $O(k)^*$ | $O(f)^*$ |
| Lower Bound | $\Omega(\log^{\epsilon}(n))$[2] | $\Omega(k^{\epsilon})$[2] | $\Omega(f^{\epsilon})$[3] |

Note: Our results are marked *. For the results of bounded set size (BSS), $k$ is the maximum number of types each agent is interested in, i.e., $|S_i| \leqslant k$. For the results of bounded frequency (BF), $f$ is the maximum number of agents interested in a particular type of item, i.e., $|\{i \in [n] \mid j \in S_i\}| \leqslant f$.

*Related Works.* The revenue that is given by optimal envy-free pricing was used as a benchmark to study the performance of truthful mechanisms where the valuations of players are not known to the mechanism[6-7]. The UD problem was shown to be APX-hard[1]; the instance assumes that each consumer is interested in a maximum of two items. Given a constant number of types of items, Hartline and Koltun[8] gave very efficient FPTASes (fully polynominal-time approximation schemes) for UD and for the unlimited supplied SM. Chen *et al.*[9] gave an optimal algorithm for selling one item in a network with unlimited supply when each UD consumer's valuations on the item selling in different nodes of the network are determined by an underlying metric. The unlimited supplied SM when all of the bundles have a limited size was studied in [4-5]. Cheung and Swamy[10] studied SM with the supply constrain that each item can only have limited number of copies. They obtain approximate solutions based on an approximation oracle for the social-welfare-maximization, i.e., the revenue maximization which does not respect the envy-freeness.

## 2 Preliminaries

### 2.1 Envy-Free Pricing

We provide an quick overview of the definition of envy-free pricing for the unit demand consumers case which this paper will focus on. For the general definition of envy-free pricing, see [1].

We assume that there are $n$ consumers and $m$ distinct types of items. A consumer $i \in [n]$ has a valuation function $v_i(\cdot)$ over all possible subsets of the items. A unit-demand consumer is interested in acquiring only a single (copy of) item.

In particular, let $\boldsymbol{p} = \langle p_1, p_2, \ldots, p_m \rangle$ be the pricing vector for the items. Let vector $\boldsymbol{A} = \langle A_1, A_2, \ldots, A_n\rangle$, where $A_i \subseteq [m]$ for all $i$, be the allocation of items to the consumers. Since we only consider unit demand consumers, we can assume $|A_i| \leqslant 1$. For simplicity, we define the utility of agent $i$ with allocation $A_i$ as

$$u_i(A_i, \boldsymbol{p}) = \begin{cases} \max_{k\in A_i} v_{i,k} - p_k, & \text{if } A_i \neq \varnothing, \\ 0, & \text{otherwise.} \end{cases}$$

**Definition 1** (Unit Demand Envy-Freeness). *A pair of pricing and allocation vectors* $(\boldsymbol{p}, \boldsymbol{A})$ *is envy-free for unit demand consumers if:*

$$u_i(A_i, \boldsymbol{p}) \geqslant \max_{j\in[m]} v_{i,j} - p_j.$$

In other words, achieving envy-freeness requires that we allocate (if any) an item to him/her that maximizes (positive) his/her utility under the pricing scheme.

For unit demand consumers, we assume $|A_i| = 1$ for all consumers. For a pair of pricing and allocation scheme $(\boldsymbol{p}, \boldsymbol{A})$, we define the revenue of the seller as

$$\sum_{i\in[n]}\sum_{k\in A_i} p_k.$$

We focus on the revenue maximization problem when the pricing and allocation are envy-free and the allocation satisfies the independent system, which we will define in Subsection 2.2. We will restrict our concern to the unit-demand consumers case.

### 2.2 Supply Constraints

Before the seller can sell the items, he/she has to produce them with resources. Different types of items require different amount of resources. Therefore, the set of total items in the market is constrained by resources. One simple constraint may be that the seller can supply at most $c_j$ copies of item $j$. In general, the constraints could be more complicated and we can express them as an independent system. In particular, let $M$ be the set of items which contains $n$ copies of each distinct type of item $j \in [m]$. We take $M$ as the ground set, which is justified since each type of item can only be allocated to at most $n$ consumers. (We treat $M$ as a set instead of a multi-set to simplify the notations.)

**Definition 2** (Independence System). $(M, \mathcal{I})$, *where* $\mathcal{I}$ *is a collection of subsets of* $M$*, is an independence system if:*

1) $\varnothing \in \mathcal{I}$.

2) *If* $B \in \mathcal{I}$ *and* $A \subseteq B$, $A \in \mathcal{I}$.

*Remark.* Sometimes we want to use multisets of $[m]$ instead of $M$. A multiset $J$ of $[m]$ is independent in $(M, \mathcal{I})$ if there exists $I \in \mathcal{I}$ such that for each $j \in [m]$, the number of items of type $j$ in $I$ is the same to $J$.

Our formulation of the constraints as an independent system is very general. It indeed captures a variety of scenarios, in addition to the simplest constraint that each item $j$ can be supplied in a possible maximum quantity of $c_j$ copies. A slightly more complicated example is to assume there are $K$ types of resources that can be used to produce the items and that we have a limited amount of $r_k$ for each resource $k \in [K]$. In order to produce one copy of item $j$, assume that we need $b_{jk}$ amount of resource $k$. Then an allocation which uses $x_j$ copies of item $j$ in total, is feasible iff the following constraints are satisfied:

$$\sum_{j\in[m]} b_{jk}x_j \leqslant r_k, \quad \text{for all} \quad k = 1, 2, \ldots, K. \qquad (1)$$

This exactly corresponds to a multi-dimensional knapsack constraint, which is an independence system. Our formulation can also express more complicated combinatorial constraints such as matroid constraints. An independence system is known as a matroid if it satisfies another property that if $X, Y \in \mathcal{I}$ and $|X| < |Y|$, then $\exists\, y \in Y \setminus X$ such that $X \cup \{y\} \in \mathcal{I}$. The independence system can also be the intersection of a number of matroids or linear constraints.

*Approximation Oracle.* In our paper, we assume the availability of an $\alpha$-approximation oracle for finding the maximum weight independent set in $\mathcal{I}$, when the weights are given. More concretely, when we assign a weight to each item, the oracle outputs an independent set in $\mathcal{I}$ whose total weight is at least $1/\alpha$ of that of the maximum weighted independent set in $\mathcal{I}$. (For different copies of the same type of item, we may assign different weights.)

Notice that, approximating the envy-free pricing problem under $\mathcal{I}$ is at least as hard as to approximate the maximum weight independent set in $\mathcal{I}$. Therefore, we are interested in settings such that good approximation algorithms exist.

For technical reasons, depending on the particular problem, we will require an oracle to compute an $\alpha$-approximate maximum weighted set in $\mathcal{I} \cap \mathcal{I}'$, where $\mathcal{I}'$ is an additional matroid defined on $[m]$. Here $\mathcal{I}'$ can be a partition matroid or a transversal matroid. A partition matroid bounds the number of items that can be picked from each group of items, where the groups are a partition of the items. A transversal matroid is defined by a bipartite graph. The ground set is the nodes in the left. A set of left nodes is independent in the transversal matroid if and only if they are covered by a matching in the bipartite graph. Although $\mathcal{I}'$ is defined on $[m]$, we can naturally extend this matroid to elements in $M$, i.e., a subset $S$ of $M$ is in $\mathcal{I}'$ if for each type of item, $S$ contains at most one copy and the set of types of items in $S$ is independent in $\mathcal{I}'$.

We remark that introducing the approximation oracle involves generalizing the supply constraints. In many interesting cases, even after adding an additional partition matroid or transversal matroid constraint, good approximation algorithms are available. For example, in the case of the knapsack constraint defined in Subsection 2.2, a PTAS (polynomial-time approximation scheme) is known when $K$ is a constant[11]. In the case of the intersection of any two matroids, an optimal algorithm exists[12]. When $\mathcal{I}$ is the intersection of $y$ matroids, an $O(y)$-approximation exists[13]. In the case of $x$ linear constraints and $y$ matroids, an $O(x+y)$ approximation appears to follow from some recent results; when $y=2$, a PTAS is recently given[14].

### 2.3 Walrasian Equilibria

It is known that Walrasian equilibria[15] are closely related to the envy-free pricing for unit demand consumers. As discussed above, let $M$ be the set of items where each type of items appears $n$ times. An allocation $\boldsymbol{A}$ can then be expressed as a matching between $M$ and $N$, because each consumer acquires at most one item. We will use a matching $W$ instead of $\boldsymbol{A}$ in most cases.

Now let $N$ be the set of consumers and $M$ be the set of items for sell. A pair of pricing and matching $(\boldsymbol{p}, W)$ is said to be a Walrasian equilibrium if it is envy-free and if any unallocated item is priced at 0. (Here we are allowed to price the copies of each type of items distinctly.) As pointed out in [1], Gul and Stacchetti's result characterizes Walrasian equilibria in unit demand case.

We define the bipartite graph $G(N \cup M, E)$ as follows. For each $i \in N$ and $j \in M$, the weight of edge $(i,j)$ is the valuation for consumer $i$ on item $j$. In particular, if item $j$ is of type $k \in [m]$, the weight of $(i,j) = v_{i,k}$. Let $MWM(X,Y)$ denote a maximum weight matching on the subgraph of $G$, induced on $X \cup Y$, where $X \subseteq N$ and $Y \subseteq M$. Notation-wise, if there is no confusion, we allow $MWM(X,Y)$ to denote the weight of the matching as well. Furthermore, they gave an algorithm that computes the Walrasian equilibrium with the highest prices.

**Theorem 1**[3]**.** *Let $(\boldsymbol{p}, W)$ be a Walrasian equilibrium. Then $W$ is a maximum weight matching in $G$, i.e., $W = MWM(N,M)$. Furthermore, for any maximum weight matching $W'$ in $G$, $(\boldsymbol{p}, W')$ is also a Walrasian equilibrium.*

We note that in any example of envy-free pricing, all allocated copies of the same item have the same price. Note that it is the case with the output of MaxWEQ.

**Theorem 2**[3]**.** *The algorithm MaxWEQ outputs, in polynomial time, a Walrasian equilibrium which maximizes the item prices. That is, for any pricing $\boldsymbol{p}$ of a Walrasian equilibrium, we have $p_j \leqslant \hat{p}_j$ for every item $j$.*

**Algorithm** MaxWEQ
**Input**: Weighted graph $G = (N \cup M, E)$.
**For** each $j \in M$, let
$\hat{p}_j = MWM(N,M) - MWM(N, M \setminus \{j\})$.
**Output**: $\hat{\boldsymbol{p}}$ and $MWM(N,M)$.

In [1], Guruswami *et al.* defined a Walrasian equilibrium with reserved prices, and showed how to compute it by reducing the problem to computing a Walrasian equilibrium.

**Definition 3**[1]**.** *Given a valuation $v_{ij}$ and a reserve price vector $\boldsymbol{r} = (r_1, \ldots, r_m)$, a Walrasian equilibrium with reserve prices $\boldsymbol{r}$ is an envy-free pricing $\boldsymbol{p}$ and allocation $W$ such that 1) $p_j \geqslant r_j$ for all $j$, 2) if item $j$ is not sold, then $p_j = r_j$, and 3) if item $j$ gives positive utility to consumer $i$ and $j$ is not sold, then consumer $i$ is allocated an item.*

**Theorem 3**[1]**.** *There exists an algorithm computing a Walrasian equilibrium with reserved prices $\boldsymbol{r}$.*

We will use these results to obtain an $O(\alpha \log n)$-approximation for the unit demand case with the general supply constraints.

## 3 General Unit Demand Case

This section will consider the general unit demand problem with general supply constraints (UD-C). Our algorithm Alg-UD-C assumes that an $\alpha$-approximation is available for finding the maximum size independent set constrained on the given independence system $\mathcal{I}$ and also on any transversal matroid defined over the items $M$. Formally, for any $E' \subseteq E$ in the bipartite graph $G(N \cup M, E)$, let $\mathcal{I}'(E')$ denote the collection of subsets of items that can be covered by a matching from $E'$. Then the oracle ($\alpha$-approximation) outputs $Y \in \mathcal{I} \cap \mathcal{I}'(E')$ such that $|Y| \geqslant \frac{1}{\alpha} \max_{Z \in \mathcal{I} \cap \mathcal{I}'(E')} |Z|$. Our algorithm is inspired by the $O(\log n)$-approximation for the problem without general supply constraints given in [1]. The analysis is similar to the proof of Lemma 3.1 in [1].

**Theorem 4.** *Suppose that we are given an $\alpha$-approximation for finding the maximum independent set that is constrained by $\mathcal{I}$ and a transversal matroid on $M$. Then Alg-UD-C is an $O(\alpha \log n)$-approximation for the UD-C problem.*

*Proof.* Let $W^*$ be a maximum weight matching of $G$ such that $Y(W^*) \in \mathcal{I}$, where $Y(W^*) \subset M$ denotes the items that are matched by $W^*$. Note that the optimal revenue is bounded by the weight of the matching, $w(W^*)$. Let $\lambda_1^*, \lambda_2^*, \ldots, \lambda_{|W^*|}^*$ be the weights of the edges

**Algorithm** Alg-UD-C for UD-C

**Input**: Weighted graph $G(N \cup M, E)$ and $(M, \mathcal{I})$.

Consider the weight of any edge $\lambda$.

Let $E'(\lambda) = \{(i,j) \in E \mid w_{ij} \geqslant \lambda\}$.

Let $\mathcal{I}'(\lambda) = \mathcal{I}'(E'(\lambda))$ denote a transversal matroid defined by $E'(\lambda)$ on $M$.

Let $B(\lambda)$ denote a set in $\mathcal{I} \cap \mathcal{I}'(\lambda)$ of the maximum size (within a factor of $\alpha$).

Set the reserve prices $\boldsymbol{r} = (r_1, r_2, \ldots, r_m)$:

**if** $j \in B(\lambda)$ **then** $r_j = \lambda$; **else** $r_j = \infty$.

Let $(\boldsymbol{p}(\lambda), W(\lambda))$ be a Walrasian equilibrium with reserve prices $\boldsymbol{r}$.

**Output**: Pair $(\boldsymbol{p}(\lambda), W(\lambda))$ for any $\lambda \geqslant 0$ with the maximum profit.

in $W^*$ in non-increasing order.

Consider when the algorithm Alg-UD-C tries $\lambda = \lambda^*_\ell$ for $\ell \in [|W^*|]$. Observe that $W^*(\lambda)$, the edges of weight at least $\lambda$ in $W^*$ is a matching and $Y(W^*(\lambda)) \in \mathcal{I}$, where $Y(W^*(\lambda))$ are the items that are matched by $W^*(\lambda)$. Thus $Y(W^*(\lambda)) \in \mathcal{I} \cap \mathcal{I}'(\lambda)$. Since $|Y(W^*(\lambda))| \geqslant \ell$, via the $\alpha$-approximation oracle, the algorithm finds $B(\lambda)$ of size at least $\ell/\alpha$. Note that the oracle attempts to find the maximum unweighted independent set.

Let $N$ denote the matching that covers $B(\lambda)$. We now show that the Walrasian equilibrium with reserve prices $\boldsymbol{r}$, $(\boldsymbol{p}(\lambda), W(\lambda))$ gives a profit of at least $\frac{\ell}{2\alpha}\lambda$. Refer to Definition 3. Consider any edge $(i,j)$ in $N$. Suppose item $j$ is not matched. Then we have $p_j = \lambda$. In this case, $i$ has positive utility for item $j$. Therefore, $i$ must be allocated with an item. Since for any edge $(i,j) \in N$, either $i$ or $j$ is matched by $W(\lambda)$ and all items are priced at least $\lambda$, the profit from $(\boldsymbol{p}(\lambda), W(\lambda))$ is at least $\frac{|W(\lambda)|}{2}\lambda \geqslant \frac{\ell}{2\alpha}\lambda (= \frac{\ell}{2\alpha}\lambda^*_\ell)$.

We are now ready to complete the analysis. Let ALG denote the algorithm's profit and OPT denote the optimal profit. For any $\ell$, we have shown that ALG $\geqslant \frac{\ell}{2\alpha}\lambda^*_\ell$. Hence it follows that OPT $\leqslant \sum_{\ell=1}^{|W^*|} \lambda^*_\ell \leqslant \sum_{\ell=1}^{|W^*|} \frac{2\alpha}{\ell}$ALG $\leqslant 2\alpha \ln n \cdot$ ALG. □

## 4 Bounded Set Size (UD-BSS)

In this section, we consider the special case that $S_i \leqslant k$ for all $i \in [n]$. We will use a relaxed version the oracle used in the previous section. In particular, we assume the availability of an $\alpha$-approximation oracle for the maximum weight independent set in $\mathcal{I}$. In other words, we do not require a transversal matroid as in the previous section.

Before we present our algorithm for UD-BSS, we first consider a general problem with independent revenue functions over items.

**Definition 4** (Maximum Production Vector). *Let $[m]$ be the set of items and $(M, \mathcal{I})$ be an independence system. Let $f_j(\cdot) : \mathbb{N} \cup \{0\} \to R^+$ be a function defined for each $j \in [m]$ satisfying $\frac{f_j(\ell)}{\ell} \geqslant \frac{f_j(\ell+1)}{\ell+1}$ for all $\ell \geqslant 1$ and $f_j(0) = 0$. The maximum production vector asks the vector $\langle a_1, a_2, \ldots, a_m \rangle$ with $a_j \in \mathbb{N} \cup \{0\}$, such that $\sum_{j \in [m]} f_j(a_j)$ is maximized, and the production vector satisfies $\mathcal{I}$.*

The maximum production vector problem captures the setting where each consumer is interested in a single item. In particular, for item $j$, let $q_1, q_2, \ldots, q_\ell$ be the set of consumers interested in $j$ with valuation $r_1 \geqslant r_2 \geqslant \cdots \geqslant r_\ell$. We can define $f_j(x) = x r_x$ for $x \in [\ell]$ and $f_j(x) = \ell r_\ell$ if $x > \ell$. The maximum production vector will give us the maximum revenue in this setting, since envy-freeness is automatically enforced when each consumer is interested in only a single item.

**Theorem 5.** *Given an $\alpha$-approximation oracle for the maximum weighted independent set of $\mathcal{I}$, the maximum production vector problem can be approximated within a factor of $2\alpha$.*

*Proof.* We first prove this result when $f_j(\cdot)$ are monotone functions. Afterwards, we show that assuming the monotonicity is without loss of generality.

For each type of item $j$, we build elements for the maximum weighted independent set instance with following weights: $\{f_j(1), f_j(2) - f_j(1), \ldots, f_j(\ell_j) - f_j(\ell_j - 1)\}$. Clearly, the total weight of the maximum weight independent set is an upper bound for the maximum production vector problem. It is then sufficient to show that an $\alpha$-approximation for the maximum weight independent set can be translated into a solution of the maximum production vector problem without much loss on the weight.

Now assume we have an independent set $I$ which is an $\alpha$-approximation for the maximum weight. We focus on a fixed item $j$ and use $f$ instead of $f_j$ to simplify the notation. In particular, let $f(j_1), f(j_2) - f(j_2 - 1), \ldots, f(j_k) - f(j_k - 1)$ be the weights of the elements of type $j$ that $I$ takes. We show

$$\sum_{x=1}^{k} (f(j_x) - f(j_x - 1)) \leqslant 2f(k). \tag{2}$$

For all $j_x \leqslant k$, the summation on the left is at most $f_j(k)$. For all $j_x > k$, we have $f(k)/k \geqslant f(j_x)/j_x \geqslant f(j_x) - f(j_x - 1)$, since $f(j_x)/j_x \leqslant f(j_x - 1)/(j_x - 1)$. Therefore, the summation of the left part of (2) for $j_x > k$ is at most $f(k)$ as there are at most $k$ terms. By setting the production of item $j$ as $k$, we obtain a production vector with weight at least half of $I$. The theorem follows.

Now consider the case that $f(\cdot)$ is not monotone. We smooth $f(\cdot)$ by introducing another monotone function $f'(\cdot)$ as follows. Let $f'(l) = \max_{l' \leqslant l} f(l')$. We now move back to explicitly using $f_j(\cdot)$. Since $f'_j(\cdot) \geqslant f_j(\cdot)$ for all possible inputs, the value of maximum production vector of $f'_j(\cdot)$s should be an upper bound for that of $f_j(\cdot)$s. On the other hand, if we have a production vector $\boldsymbol{v}'$ for $f'_j(\cdot)$, we can build another vector $\boldsymbol{v}$ for $f_j(\cdot)$s having the same value by decreasing the entries accordingly, i.e., if the $j$-th entry $v'_j \in \boldsymbol{v}'$ such that $f'_j(v'_j) > f_j(v'_j)$, we simply find another value $v_j < v'_j$ with $f_j(v_j) = f'_j(v'_j)$ by our construction of $f'_j(\cdot)$. Because of the downward closure property of $\mathcal{I}$, the new production vector $\boldsymbol{v}$ is also independent. Therefore, an approximate maximum production vector for $f'_j(\cdot)$ can be converted to a vector for $f_j(\cdot)$ which preserves the approximation ratio. Assuming the monotonicity of $f_j(\cdot)$ is thus without loss of generality. □

**Corollary 1.** *Given an $\alpha$-approximation oracle for the maximum weighted independent set in $\mathcal{I}$, one can obtain a $2\alpha$-approximation for UD-BSS-C with $k = 1$.*

The following pseudocode is our algorithm for UD-BSS-C. Following the randomized partition in [4], for each item type, we add it to $Y$ with probability $1/k$. We then consider only those consumers (i.e., $X$) who are interested in a single type of item in $Y$, in which case Corollary 1 can be applied.

**Algorithm** Alg-UD-BSS-C for UD-BSS-C
**Input**: $[n], [m], v_{ij}$.
Let $Y \leftarrow \varnothing$.
Add each $j \in [m]$ to $Y$ with independent probability $1/k$.
Define $X = \{i \in [n] \mid |Y \cap S_i| = 1\}$.
Compute optimal envy-free pricing $(\boldsymbol{p}', W')$ for $(X, Y)$.
Let $Y'$ be the items matched by $M'$.
Let $(\boldsymbol{p}, W) = \text{MaxWEQ}([n], Y')$.
**For** all $j \in [m] \setminus Y'$, $p_j = \infty$.
**Output**: $(\boldsymbol{p}, M)$.

The pricing strategy output by Corollary 1 is an envy-free pricing scheme if only the consumers in $X$ exist in the market. The next lemma shows that adding more customers, thus introducing more demands, does not decrease the revenue under the envy-free condition.

**Lemma 1.** *Consider a subset of consumers $X \subseteq [n]$. Let $\boldsymbol{p}$ be an envy-free pricing for $(X, Y)$, where all items in $Y$ are assigned. One can compute a Walrasian equilibrium (thus envy-free) of pricing $\boldsymbol{p}'$ for $([n], Y)$ such that $\boldsymbol{p}' \geqslant \boldsymbol{p}$, i.e., $p'_j \geqslant p_j$ in polynomial time. Furthermore, the total revenue of the Walrasian equilibrium in $([n], Y)$ is at least the revenue of $\boldsymbol{p}$ in $(X, Y)$.*

*Proof.* For each item $j \in Y$, we set the price $p'_j = MWM([n], Y) - MWM([n], Y \setminus \{j\})$. By MaxWEQ, these prices form a Walrasian equilibrium. We first show that $p'_j \geqslant p_j$.

Let $W$ be the matching of an envy-free assignment for $(X, [m])$ with prices $\boldsymbol{p}$. Without loss of generality, we consider item 1 and price $p'_1$. Let denote $W' = MWM([n], [m] \setminus \{1\})$. It is sufficient to show that when we add item 1, there exists a way to improve the maximum weight matching by at least $p_1$.

Let $i_1 \in A$ be the consumer matched with item 1 in $W$. Now if $i_1$ is not matched in $W'$, we can simply add $(i_1, 1)$ to $W'$. Therefore, $p'_1 \geqslant v_{i_1,1} \geqslant p_1$, where $v_{i,j}$ is the valuation of consumer $i$ on item $j$.

Define $j_1 = 1$. If $i_1$ is matched with another item $j_2$ in $W'$, we can find a sequence of items $\{j_1 = 1, j_2, \ldots, j_\ell\}$ and consumers $\{i_1, i_2, \ldots, i_\ell\}$ such that: 1) for all $1 \leqslant l \leqslant \ell$, $i_l$ is matched with $j_l$ in $W$; 2) for all $1 \leqslant l \leqslant \ell - 1$, $i_l$ is matched with $j_{l+1}$ in $W'$; 3) $i_\ell$ is unmatched in $W'$. We obtain a new matching by replacing the edges of $(i_l, j_{l+1})$ for $1 \leqslant l \leqslant \ell - 1$ in $W'$ with the edges $(i_l, j_l)$ for $1 \leqslant l \leqslant \ell$.

Now we consider the change of total weights. Since $W$ is an envy-free assignment, $v_{i_l,j_{l+1}} - p_{j_{l+1}} \leqslant v_{i_l,j_l} - p_{j_l}$ for all $l \in [\ell - 1]$. So we have

$$\sum_{l=1}^{\ell-1}\Big(v_{i_l,j_{l+1}} - p_{j_{l+1}}\Big) \leqslant \sum_{l=1}^{\ell-1}(v_{i_l,j_l} - p_{j_l})$$
$$\Rightarrow \sum_{l=1}^{\ell-1} v_{i_l,j_{l+1}} \leqslant \Big(\sum_{l=1}^{\ell-1} v_{i_l,j_l}\Big) - p_{i_1} + p_{i_l}$$
$$\leqslant \Big(\sum_{l=1}^{\ell} v_{i_l,j_l}\Big) - p_{i_1}.$$

The last inequality comes from the fact that $v_{i_\ell,j_\ell} \geqslant p_{i_\ell}$. Therefore, $p'_1 \geqslant p_{i_1} = p_1$.

Now consider the overall revenue. Notice in a Walrasian equilibrium, all items with positive prices are allocated. The total revenue is at least $\sum_{i\in[m]} p'_i \geqslant \sum_{i\in[m]} p_i$. □

**Theorem 6.** *Given an $\alpha$-approximation oracle for the maximum weight independent set for $\mathcal{I}$, when $|S_i| \leqslant k$ for all $i$, there exists a $(2\alpha ek)$-approximation for UD-BSS-C.*

*Proof.* Consider the optimal revenue for the market $(X, Y)$ with the constraint of $\mathcal{I}$ in Alg-UD-BSS-C. Notice that each consumer in $X$ has exactly one item of interest in $Y$. Corollary 1 claims that we can obtain a $2\alpha$-approximate optimal revenue for the market $(X, Y)$. Hence, it is sufficient to show that the expected optimal revenue of the market $(X, Y)$ is at least $1/(ek)$ of the optimal revenue of the original instance for UD-BSS-C. Notice that adding more consumers does not decrease the revenue by Lemma 1; adding items into the market also does not decrease the revenue since one can simply set their prices arbitrarily high.

We now give a pricing scheme $\boldsymbol{p}'$ that gives an expected revenue of at least $\frac{1}{ek}$OPT for the market $(X, Y)$. Let $\boldsymbol{p}^*$ be the optimal solution's pricing and $C_j^*$ the set of consumers that are assigned item $j$ in the optimal solution. Note that the optimal revenue can be written as $\text{OPT} = \sum_j p_j^* |C_j^*|$. To have a feel of how the pricing scheme works, consider the following pricing $\boldsymbol{p}'$: for each $j$, if $j \in Y$ then $p_j' = p_j^*$, otherwise $p_j' = \infty$. Then some consumers that were allocated items in $[m] \setminus Y$ may move to items in $Y$, thereby making some items in $Y$ out of stock. To prevent it, we increase the price of each item $j$ that is over demanded in the market $(X, Y)$ under the pricing $\boldsymbol{p}'$. This is well justified since the effect of increasing item $j$'s price is restricted to the consumers that are interested in the item $j$, and each consumer in $X$ is interested in only one item in $Y$. Thus we restrict our attention to the consumers in $C_j^*$. By setting item $j$'s price at a proper level of at least $p_j^*$, we can obtain a revenue of at least $p_j^*|\{i \in C_j^* \mid i \in X \text{ and } j \in Y\}|$. Therefore the expected revenue we obtain by selling item $j$ is at least

$$\begin{aligned} & p_j^* \mathbb{E}[|\{i \in C_j | i \in X \ \wedge \ j \in Y\}|] \\ & = p_j^* \sum_{i \in C_j^*} Pr[i \in X \wedge j \in Y] \\ & \geqslant p_j^* \frac{1}{k}\Big(1 - \frac{1}{k}\Big)^{k-1} |C_j^*| \geqslant \frac{p_j^* |C_j^*|}{ek}. \end{aligned}$$

By linearity of expectation, the lemma easily follows. Finally, we give the pricing $\boldsymbol{p}$ that serves for the above argument. Let $T_j = \{i \in [n] \mid j \in S_i\}$ denote the consumers that are interested in item $j$. Let $p_j' = \arg\max_p \Big(|\{i \in T_j \cap X \mid w_{i,j} \geqslant p\}| \geqslant |C_j^*|\Big)$. We set the price of $j$ as $p_j = \max\{p_j', p_j^*\}$, i.e., the price is at least $p_j^*$ while the demand does not exceed $|C_j^*|$. For each item $j \notin Y$, we set $p_j = \infty$. □

## 5 Bounded Frequency (UD-BF)

This section addresses the problem UD-BF, in which each item is of interest to at most $f$ consumers. Let $\{G_1, G_2, \ldots, G_\ell\}$ denote an arbitrary partition of items $M$, and $\mu_l \geqslant 0$ be an integer that is associated with $G_l$ for $l \in [\ell]$. A collection of sets of items $\mathcal{I}' \subseteq 2^M$, with items $M$, is said to be a partition matroid if $X \in \mathcal{I}'$ if and only if for all $l \in [\ell]$, $|X \cap G_l| \leqslant \mu_l$. In other words, an independent set can only pick $\mu_l$ elements from group $G_l$.

We assume that for any partition matroid $(M, \mathcal{I}')$ (for our purposes, we only need $\mu_l \in \{0, 1\}$ for all $l$), we can use an $\alpha$-approximation for finding the maximum weight independence set constrained on $\mathcal{I}$ and $\mathcal{I}'$. We give the following randomized algorithm.

Using a random sampling, the algorithm Alg-UD-BF-C recasts the given instance to the market $(X, Y)$ where each item in $Y$ is interesting only to a single consumer in $X$. This creates a natural partition matroid on $[m]$, as well as on $M$. With the aid of the $\alpha$-approximation, the algorithm outputs the desired result.

**Algorithm** Alg-UD-BF-C for UD-BF-C
**Input**: $[n], [m], v_{ij}, \mathcal{I}$.
Let $X \leftarrow \varnothing$.
Add each $i \in [n]$ to $X$ with independent probability $1/f$.
Let $Y := \{j \in [m] \mid |\{i \in [n] \mid j \in S_i\}| = 1\}$.
$\forall\, i \in X$, assign weight $v_{ij}$ to each $j \in S_i \cap Y$.
Let $\mathcal{I}' = \{B \subseteq [m] \mid \forall i \in X, |S_i \cap B \cap Y| \leqslant 1\}$.
Compute an $\alpha$-approximation of the maximum subset of $Y$ in $\mathcal{I} \cap \mathcal{I}'$.
**Output**: $(\boldsymbol{p}, M) \leftarrow \text{MaxWEQ}([n], Y')$, for any $j \notin Y'$, $p_j = \infty$.

**Theorem 7.** *Consider any partition matroid $(M, \mathcal{I}')$. Suppose that we have an $\alpha$-approximation for the maximum weight independent set constrained on $\mathcal{I}$ and also $\mathcal{I}'$. Then the randomized algorithm Alg-UD-BF-C outputs an envy-free pricing that gives an expected revenue that is optimal within a factor of $\alpha e f$.*

*Proof.* Let $W^*$ denote the matching (allocation) in a fixed optimal solution. Let $N^*$ denote the edges in $W^*$ that appear in the market $(X, Y)$, i.e., $N^* = W^* \cap (X \times Y)$. It is not difficult to see that the set of the items matched by $N^*$ is in $\mathcal{I} \cap \mathcal{I}'$. For any $(i, j) \in W^*$, the event that $(i, j) \in N^*$ occurs with a probability of at least $\frac{1}{f}(1 - \frac{1}{f})^{f-1} \geqslant \frac{1}{ef}$. Thus $\mathbb{E}[w(N^*)] \geqslant \frac{1}{ef}$OPT, where $w(N^*)$ denotes the total weight of edges in $N^*$. Using the $\alpha$-approximation for the maximum weight independence set approximation constrained on $\mathcal{I} \cap \mathcal{I}'$, the algorithm obtains $Y'$ whose total weight is at least $\frac{1}{\alpha} w(N^*)$; thus the expected value is at least $\frac{1}{\alpha e f}$OPT. By definition of $\mathcal{I}'$, we have a natural matching that each item in $Y$ is matched with the unique consumer in $X$ that is interested in the item. For each matched item $j$ in $Y$, we give it the price of the weight of the edge that covers $j$ in the matching; unmatched items are priced at $\infty$. Then it is easy to see that such a pricing and matching is envy-free in the market $(X, Y')$, has an expected revenue of at least $\frac{1}{\alpha e f}$OPT and all items in $Y'$ are allocated. By Lemma 1, the final output of the algorithm, which invites more consumers to the market, achieves no smaller revenue; here all items not in $Y'$ are priced at $+\infty$. □

## 6 Conclusions

In this paper, we initiated a study on the envy-free pricing problem with general supply constraints for unit demand consumers. Cheung and Swamy[10] studied the envy-free pricing problem with single-minded consumers under supply constrains. However, they have more restricted condition on the constrains. It is an interesting problem to extend the more general supply constrains to single-minded consumers.

We note that all of our algorithms can be derandomized in polynomial time as done in [4] using the technique of approximating the joint distribution of random variables with a small probability space[16].

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