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An FPTAS for the hardcore model on random regular bipartite graphs

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A B S T R A C T

We give a fully polynomial-time approximation scheme (FPTAS) to compute the partition function of the hardcore model of fugacity λ on random Δ-regular bipartite graphs for all sufficiently large Δ and $\lambda \geq 4(\log\Delta)^3/\Delta$. For the special case of $\lambda = 1$, where the partition function computes the number of independent sets, an FPTAS exists for $\Delta \geq 50$. Our technique is based on the polymer model, which is used by Jenssen, Keevash and Perkins (SODA, 2019) to obtain an FPTAS for #BIS-hard problems for the first time. The technique also applies to counting q-colorings: For $q \geq 3$ and $\Delta \geq \Delta(q)$, there is an FPTAS to compute the number of q-colorings on random Δ-regular bipartite graphs.

1. Introduction

For a graph G and a parameter (called fugacity or activity) $\lambda > 0$, the hardcore model is the Gibbs measure on independent sets $\mathcal{I}(G)$

$$\mu(I) = \lambda^{|I|}/Z(G,\lambda),$$

where the normalizing factor

$$Z(G,\lambda) = \sum_{I\in\mathcal{I}(G)} \lambda^{|I|}$$

is called the partition function. The problems of evaluating the sum and sampling an independent set from the distribution have been extensively studied in computer science, discrete probability and statistical physics. In this paper, we focus on the approximation of the partition function.

In a seminal paper, Weitz [1] first presented a fully polynomial-time approximation scheme (FPTAS) for the partition function on graphs of maximum degree Δ when $\lambda < \lambda_c(\Delta) = \frac{(\Delta-1)^{\Delta-1}}{(\Delta-2)^{\Delta}}$. The quantity $\lambda_c(\Delta)$ is the uniqueness threshold of the Gibbs measure on the infinite Δ-regular tree. On the hardness side, Sly [2] proved that, for $\lambda_c(\Delta) < \lambda < \lambda_c(\Delta) + \varepsilon(\Delta)$, no polynomial-time approximation scheme exists unless $\mathbf{NP} = \mathbf{RP}$. Later, this result was improved to any $\lambda > \lambda_c(\Delta)$ [3,4]. In

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particular, these results state that there is an FPTAS for counting independent sets ($\lambda = 1$) on graphs of maximum degree $\Delta \le 5$, and the approximation is NP-hard if $\Delta \ge 6$.

However, no NP-hardness result is known for counting independent sets on bipartite graphs (#BIS). Liu and Lu [5] designed an FPTAS for #BIS which requires only vertices from one partition to be of maximum degree $\Delta \le 5$. For bipartite graphs of maximum degree $\Delta \ge 3$ and $\lambda > \lambda_c(\Delta)$, it is #BIS-hard to approximate the partition function of the hardcore model [6]. The problem #BIS is conjectured to be of intermediate complexity [7]. In fact, a wide range of counting problems in the study of counting CSPs [8–10] and spin systems [11–13,6], have been proved to be #BIS-equivalent or #BIS-hard under approximation-preserving reductions (AP-reductions).

Recently, Helmuth, Perkins, and Regts [14] developed a new approach via the polymer model and gave efficient counting and sampling algorithms for the hardcore model at high fugacity on certain finite regions of the lattice $\mathbb{Z}^d$ and on the torus $(\mathbb{Z}/n\mathbb{Z})^d$. Their approach is based on a long line of work [15–20]. The polymer model gives some new insights into the hardcore model on bipartite graphs [21–23]. Jenssen, Keevash, and Perkins [21] designed an FPTAS for the high fugacity case on bipartite expander graphs of bounded degree. They further extended the result to random Δ-regular bipartite graphs with $\Delta \ge 3$ and $\lambda > (2e)^{250}$. A natural question is, can we design an FPTAS for lower fugacity and in particular the problem #BIS on random regular bipartite graphs? Indeed, we obtain such results. Let $\mathcal{G}_{n,\Delta}^{\mathrm{bip}}$ denote the set of all Δ-regular bipartite graphs with n vertices on both partitions.

Theorem 1. *For sufficiently large Δ, there is an FPTAS such that with high probability (tending to 1 as $n \to \infty$) for a graph G chosen uniformly at random from $\mathcal{G}_{n,\Delta}^{\mathrm{bip}}$ the FPTAS computes $Z(G,\lambda)$ for $\lambda \ge 4(\log \Delta)^3/\Delta$.*

Theorem 2. *For $\Delta \ge 50$, there is an FPTAS such that with high probability (tending to 1 as $n \to \infty$) for a graph G chosen uniformly at random from $\mathcal{G}_{n,\Delta}^{\mathrm{bip}}$ the FPTAS computes the number of independent sets of G.*

Counting proper q-colorings on a graph is another extensively studied problem in the field of approximate counting [24–29]. In general graphs, if the number q of colors is no more than the maximum degree Δ, there may not be any proper coloring over the graph. Therefore, approximate counting is studied in the range that $q \ge \Delta + 1$. It was conjectured that there is an FPTAS or FPRAS if $q \ge \Delta + 1$, but the current best result is $q \ge \alpha\Delta + 1$ with a constant α slightly below $\frac{11}{6}$ [30,28]. The conjecture was only confirmed for the special case $\Delta = 3$ [31].

On bipartite graphs, the situation is quite different. For any $q \ge 2$, we know that there always exist proper q-colorings for every bipartite graph. For any $q \ge 3$, it is shown to be #BIS-hard but unknown to be #BIS-interreducible [7]. Using a technique analogous to that for #BIS, we obtain an FPTAS to count the number of q-colorings on random Δ-regular bipartite graphs for sufficiently large integers $\Delta(q)$ for any $q \ge 3$.

Theorem 3. *For $q \ge 3$ and $\Delta \ge 80q^3 \log^2 q$, there is an FPTAS such that with high probability (tending to 1 as $n \to \infty$) for a graph chosen uniformly at random from $\mathcal{G}_{n,\Delta}^{\mathrm{bip}}$ the FPTAS computes the number of q-colorings of G.*

Our technique

A classical approach for approximate counting algorithms is sampling via Markov chain Monte Carlo (MCMC). However, it is shown in [32] that the Glauber dynamics for independent sets is slowly mixing on random Δ-regular bipartite graphs for $\Delta \ge 6$. A typical independent set of such a graph is unbalanced: It either chooses most of its vertices from the left partition or the right partition. Thus, starting from an independent set with most vertices from the left partition, a Markov chain is unlikely to reach an independent set with most of its vertices from the right partition in polynomial time.

The beautiful work [21] exactly makes use of the above separating property to design approximate counting algorithms. By making the fugacity sufficiently large, they proved that most contribution of the partition function comes from extremely unbalanced independent sets, those which occupy almost no vertices on one partition and almost all vertices on the other partition. In particular, for a bipartite graph $G = (\mathcal{L}, \mathcal{R}, E)$ with n vertices on both partitions, they identified two independent sets $I_1 = \mathcal{L}$ and $I_2 = \mathcal{R}$ as ground states as they have the largest weight λ^n among all the independent sets. They proved that one only needs to sum up the weights of states (independent sets) which are close to the ground states, for no state is close to both ground states and the contribution from the states which are far away from both ground states is exponentially small.

However, the idea of ground states cannot be directly applied to counting independent sets and counting colorings since each valid configuration is of the same weight. We extend the idea of ground states to ground clusters, which is not a single configuration but a family of configurations. For example, we identify two ground clusters for independent sets, those which are entirely chosen from vertices on the left partition and those which are entirely chosen from vertices on the right partition. If a set of vertices is entirely chosen from vertices on one partition, it is obviously an independent set. Thus each cluster contains 2^n different independent sets. Similarly, we want to prove that we can count the configurations which are close to one of the ground clusters and then add them up. For counting colorings, there are multiple ground clusters indexed by a subset of colors X: Colorings which map $\mathcal{L}$ and $\mathcal{R}$ to the sets X and $[q] \setminus X$, respectively.

Unlike the ground states in [21], our ground clusters may overlap with each other and some configurations are close to more than one ground clusters. In addition to proving that the number of configurations which are far away from all ground clusters are exponentially small, we also need to prove that the number of double counted configurations are small.

After identifying ground states, the authors of [21] fixed a ground state and defined a polymer model representing deviations from the ground state and rewrote the original partition function as a polymer partition function. We follow this idea and define a polymer model representing deviations from a ground cluster. However, deviation from a ground cluster is much subtler than deviation from a single ground state. For example, if we define polymer as connected components from the deviated vertices in the graph, we cannot recover the original partition function from the polymer partition function. We overcome this by defining polymer as connected components in graph G^2, where an edge of G^2 corresponds to a path of length at most 2 in the original graph. Here, a compatible set of polymers also corresponds to a family of configurations in the original problem, while it corresponds to a single configuration in [21].

It is much more common in counting problems that most contribution is from a neighborhood of some clusters rather than a few isolated states. So, we believe that our development of the technique makes it suitable for a broader family of problems.

Independent work

The journal version of [21] obtained similar results (see [33]).

2. Preliminaries

2.1. Independent sets and colorings

Let $G=(V,E)$ be a graph. For $A,B\subseteq V$, let $d_G(A,B)$ be the distance between A and B in G. For $A\subseteq V$, let $N_G(A)$ be the set of vertices of distance 1 to A. For a graph H, we use $V(H)$ to denote the set of its vertices. If H is a subgraph of G, we use $N_G(H)$ to denote $N_G(V(H))$. For $u,v\in V$, we write $d_G(u,v)$ for $d_G(\{u\},\{v\})$ and write $N_G(v)$ for $N_G(\{v\})$. We write $G[A]$ for the induced subgraph on $A\subseteq V$. We define $G^2=(V,E')$ where $(u,v)\in E'$ if $u\neq v$ and $d_G(u,v)\le 2$. Note that if G is of maximum degree Δ, then G^2 has maximum degree at most Δ^2.

A set $I\subseteq V$ is an independent set if $d_G(u,v)>1$ for any $u,v\in I$ with $u\neq v$. Let λ be a parameter and the weight of I is $\lambda^{|I|}$. We write $\mathcal{I}(G)$ for the set of all independent sets of G. Independent sets correspond to valid configurations of the hard-core model in statistical physics and the partition function of the model is $Z(G,\lambda)=\sum_{I\in\mathcal{I}(G)}\lambda^{|I|}$.

Let $[n]$ be the set $\{1,2,\dots,n\}$. For an integer $q\ge 3$, let $\underline{q}=\lfloor q/2\rfloor$ and $\overline{q}=\lceil q/2\rceil$. A coloring $\sigma\in[q]^V$ of G labels each vertex with some color in $[q]$. We say σ is proper if $\sigma(u)\neq\sigma(v)$ for any edge $(u,v)\in E$. We write $\mathcal{C}(G)$ for the set of all proper colorings of G. It will be useful to restrict the domain of a coloring and we write $\sigma_A\in[q]^A$ for the coloring that $\sigma_A(v)=\sigma(v)$ for all $v\in A$. For disjoint sets $A_1,\dots,A_k\subseteq V$ and colorings $\sigma_i\in[q]^{A_i}$, we write $\uplus_{i=1}^k\sigma_i$ for the coloring $\sigma\in[q]^{\cup_{i=1}^k A_i}$ that $\sigma(v)=\sigma_i(v)$ for any $i\in[k]$ and $v\in A_i$.

For positive real numbers a and b, we say a is an ε-relative approximation to b for $\varepsilon>0$ if $e^{-\varepsilon}\le a/b\le e^{\varepsilon}$. A fully polynomial-time deterministic approximation scheme (FPTAS) is a deterministic algorithm that for every $\varepsilon>0$ and a problem instance $\boldsymbol{I}$ it outputs an ε-relative approximation to $Z(\boldsymbol{I})$ in time $\mathrm{poly}(|\boldsymbol{I}|,1/\varepsilon)$, where $Z(\boldsymbol{I})$ is a quantity of the instance $\boldsymbol{I}$ to compute.

2.2. Random regular bipartite graphs

Let Δ be a positive integer. Let G be a bipartite graph with n vertices on both partitions and the edges of G are Δ perfect matchings of the complete bipartite graph $K_{n,n}$ chosen uniformly at random and independently. We allow multiple edges of G because it has the same effect on independent sets and colorings as single edges. Thus G would be Δ-regular and we use $G\sim\mathcal{G}^{\mathrm{bip}}_{n,\Delta}$ to denote such a random graph. This distribution would be very close to the one where a Δ-regular bipartite simple graph with n vertices on both partitions is chosen uniformly at random. It follows from [34] that Lemma 4 and other results in this paper also apply to the latter distribution.

We say a Δ-regular bipartite graph $G=(\mathcal{L},\mathcal{R},E)$ with $|\mathcal{L}|=|\mathcal{R}|=n$ is an (α,β)-expander if for all $U\subseteq\mathcal{L},V\subseteq\mathcal{R}$ with $|U|,|V|\le\alpha n$, there is $|N_G(U)|\ge\beta|U|$ and $|N_G(V)|\ge\beta|V|$. We also refer to this as the expansion property of G. Let $\mathcal{G}^{\Delta}_{\alpha,\beta}$ be the set of all Δ-regular bipartite (α,β)-expander graphs. The following lemma states that almost every Δ-regular graph is an expander.

Lemma 4 *([35]). If* $0<\alpha<1/\beta<1$ *and* $\Delta>\dfrac{H(\alpha)+H(\alpha\beta)}{H(\alpha)-\alpha\beta H(1/\beta)}$, *then*

$$\lim_{n\to\infty}\Pr_{G\sim\mathcal{G}^{\mathrm{bip}}_{n,\Delta}}\left[G\in\mathcal{G}^{\Delta}_{\alpha,\beta}\right]=1,$$

where $H(x)=-x\log x-(1-x)\log(1-x)$ *is the binary entropy function.*

2.3. Polymer models

Let G be a graph and Ω be a set of values. A polymer $\gamma = (\overline{\gamma}, \omega_{\overline{\gamma}})$ defined on G consists of a connected subgraph $\overline{\gamma}$ of G (called the support of γ) and a labeling $\omega_{\overline{\gamma}}$ which gives each vertex in $\overline{\gamma}$ a value in Ω. We use $|\overline{\gamma}|$ to denote the number of vertices in $\overline{\gamma}$. We say two polymers γ_1 and γ_2 are compatible, denoted by $\gamma_1 \sim \gamma_2$, if $d_G(\overline{\gamma_1}, \overline{\gamma_2}) > 1$. For a set Γ of polymers, it is compatible if any two different polymers in it are compatible. Let $\mathsf{cpt}(\Gamma) = \{\Gamma' \subseteq \Gamma : \Gamma' \text{ is compatible}\}$. For any $\Gamma' \in \mathsf{cpt}(\Gamma)$, let $\overline{\Gamma'}$ be the union of the support of all polymers in Γ', which is a subgraph of G. Let $\left|\overline{\Gamma'}\right|$ be the number of vertices in $\overline{\Gamma'}$. Let $\omega_{\overline{\Gamma'}}$ be the union of labellings $\omega_{\overline{\gamma}}$ of polymers $\gamma \in \Gamma'$. Let $w : \Gamma \times \mathbb{C} \to \mathbb{C}$ be a function. Then (Γ, w) is a polymer model defined on G with partition function

$$\Xi(G, z) = \sum_{\Gamma' \in \mathsf{cpt}(\Gamma)} \prod_{\gamma \in \Gamma'} w(\gamma, z).$$

The following theorem gives efficient approximation algorithms for $\Xi(G, z)$.

Theorem 5 *([14], Theorem 2.2, with suitable modification). Fix Δ and let $\mathcal{G}$ be a set of graphs of degree at most Δ. Suppose for every graph $G \in \mathcal{G}$ there is a polymer model (Γ, w) defined on G. If*

- *There is a constant C such that for all $G \in \mathcal{G}$, the degree of $\Xi(G, z)$ is at most $C|G|$.*
- *For all $G \in \mathcal{G}$ and $\gamma \in \Gamma(G)$, $w(\gamma, z) = a_\gamma z^{|\overline{\gamma}|}$ where $a_\gamma \neq 0$ can be computed in time $\exp(O(|\overline{\gamma}| + \log|G|))$.*
- *For every connected subgraph G' of every $G \in \mathcal{G}$, we can list all polymers $\gamma \in \Gamma(G)$ with $\overline{\gamma} = G'$ in time $\exp(O(|G'|))$.*
- *There is a constant $R > 0$ such that for all $G \in \mathcal{G}$ and $z \in \mathbb{C}$ with $|z| < R$, $\Xi(G, z) \neq 0$.*

Then there is an FPTAS to compute $\Xi(G, z)$ for all $G \in \mathcal{G}$ and $|z| < R$.

The following condition is useful to show that $\Xi(G, z) \neq 0$.

Lemma 6 *([17], KP-condition). Fix $z \in \mathbb{C}$. Suppose there is a function $a : \Gamma \to \mathbb{R}_{>0}$ and for every $\gamma \in \Gamma$,*

$$\sum_{\gamma' : \gamma' \nsim \gamma} e^{a(\gamma')} \left| w(\gamma', z) \right| \leq a(\gamma).$$

Then $\Xi(G, z) \neq 0$.

To verify KP-condition, the support of polymers $\gamma' \nsim \gamma$ is usually connected and induced, hence the following lemma is helpful.

Lemma 7 *([36]). For any graph $G = (V, E)$ of maximum degree Δ and $v \in V$, the number of connected induced subgraphs of $k \geq 2$ vertices containing v is at most $(e\Delta)^{k-1}/2$.*

2.4. Some useful facts

Throughout this paper, we write $\log x$ for a shorthand of $\log_2 x$. A fact that will be heavily used is that $\log(1 + x) \leq x$ for $x \in (-1, 0]$ and $\log(1 + x) \geq x$ for $x \in [0, 1]$. Sometimes we need to use a stronger version of this fact, that is, $\log(1 + x) \leq x \cdot \log e$ for $x \in (-1, 0)$. Recall that $H(x) = -x \log x - (1 - x) \log(1 - x)$ for $x \in [0, 1]$ with the convention $0 \log 0 = 0$.

Lemma 8 *([37, Lemma 10.2]). Suppose that n is a positive integer and $k \in [0, 1]$ is a number such that kn is an integer. Then*

$$\frac{2^{H(k)n}}{n+1} \leq \binom{n}{kn} \leq 2^{H(k)n}.$$

Lemma 9. *For $0 \leq x \leq 1/2$, $H(x) \leq -2x \log x$.*

Proof. Let $f(x) = H(x) + 2x \log x$. Then $f(0) = f(1/2) = 0$ and convexity of f over $[0, 1/2]$ gives the lemma. □

Lemma 10. *For $y > 1$ and $0 \leq x \leq 1/y$, $H(x) - H(xy)/y \geq x \log y$.*

Proof. Fix $y > 1$. For $0 < x < 1/y$,

$$\begin{aligned} H(x) - H(xy)/y &= x\log y - (1-x)\log(1-x) + 1/y(1-xy)\log(1-xy) \\ &\geq x\log y. \end{aligned}$$

The inequality holds because the function

$$F(x) = (1-x)\log(1-x)$$

is convex and hence $F(yx) \geq yF(x)$. $\square$

3. Counting independent sets

This section is devoted to the proof of Theorem 1. Let α be the solution to $\Delta = \frac{-4\log\alpha}{\alpha}$, $\beta = \frac{1-\alpha}{\alpha}$ and $\lambda_* = \alpha \cdot (\log\Delta)^2$ throughout this section. We verify that $\frac{1}{\alpha} \leq \Delta$ because otherwise $-4\log\alpha/\alpha > 4\Delta\log\Delta > \Delta$. Therefore $\alpha = \frac{4\log(1/\alpha)}{\Delta} \leq \frac{4\log\Delta}{\Delta}$.

Lemma 11. *For sufficiently large integers Δ, $\lim_{n\to\infty} \Pr_{G\sim\mathcal{G}^{\text{bip}}_{n,\Delta}}\left[G \in \mathcal{G}^{\Delta}_{\alpha,\beta}\right] = 1$.*

Proof. We verify the conditions in Lemma 4. Apply Lemma 10 ($y = \frac{1}{\alpha\beta} \geq 1, x = \alpha \leq 1/y$),

$$H(\alpha) - \alpha\beta H(1/\beta) \geq \alpha\log(1/\alpha\beta) = -\alpha\log(1-\alpha) > \alpha^2.$$

Then

$$\frac{H(\alpha)+H(\alpha\beta)}{H(\alpha)-\alpha\beta H(1/\beta)} < \frac{H(\alpha)+H(1-\alpha)}{\alpha^2} = \frac{2H(\alpha)}{\alpha^2} \leq \frac{-4\log\alpha}{\alpha} = \Delta,$$

where we use Lemma 9 to bound $H(\alpha)$. $\square$

In the rest of this section, if not specified, let $G = (\mathcal{L}, \mathcal{R}, E) \in \mathcal{G}^{\Delta}_{\alpha,\beta}$ be a bipartite graph with n vertices on both partitions. Moreover, we assume that $n > N$ for some sufficiently large constant $N(\Delta) > 0$.

3.1. Approximating $Z(G,\lambda)$

For $\mathcal{X} \in \{\mathcal{L}, \mathcal{R}\}$, let

$$\mathcal{I}_{\mathcal{X}}(G) = \{I \in \mathcal{I}(G) : |I \cap \mathcal{X}| \leq \alpha n\},\ Z_{\mathcal{X}}(G,\lambda) = \sum_{I\in\mathcal{I}_{\mathcal{X}}(G)} \lambda^{|I|}.$$

We show that $Z_{\mathcal{L}}(G,\lambda) + Z_{\mathcal{R}}(G,\lambda)$ is very close to $Z(G,\lambda)$.

Lemma 12. *For sufficiently large Δ and $\lambda \geq \lambda_*$, $Z_{\mathcal{L}}(G,\lambda) + Z_{\mathcal{R}}(G,\lambda)$ is a K^{-n}-relative approximation to $Z(G,\lambda)$ for some constant $K(\Delta) > 1$.*

Proof. For an independent set I that $|I \cap \mathcal{L}| > \alpha n$, there are at least $\lfloor \alpha n \rfloor$ vertices of I in $\mathcal{L}$. It follows from the expansion property that $|N(I \cap \mathcal{L})| \geq \beta\lfloor\alpha n\rfloor \geq \beta(\alpha n - 1) = (1-\alpha)n - \beta$. Thus $|I \cap \mathcal{R}| \leq n - |N(I\cap\mathcal{L})| \leq \alpha n + \beta$. By symmetry we know that if $I \notin \mathcal{I}_{\mathcal{L}} \cup \mathcal{I}_{\mathcal{R}}$ then $\alpha n < |I \cap \mathcal{X}| \leq \alpha n + \beta$ for $\mathcal{X} \in \{\mathcal{L}, \mathcal{R}\}$. Let $Z_{\text{omitted}} = \sum_{I\notin\mathcal{I}_{\mathcal{L}}\cup\mathcal{I}_{\mathcal{R}}} \lambda^{|I|}$. Since $Z(\lambda) \geq (1+\lambda)^n$,

$$\begin{aligned} \frac{Z_{\text{omitted}}(\lambda)}{Z(\lambda)} &\leq \frac{\left(\sum_{k=\lceil\alpha n\rceil}^{\lfloor\alpha n+\beta\rfloor}\binom{n}{k}\lambda^k\right)^2}{(1+\lambda)^n} \leq \frac{\left(n\cdot\max_{\alpha\leq\theta\leq\alpha+\beta/n} 2^{H(\theta)n}\lambda^{\theta n}\right)^2}{(1+\lambda)^n} \\ &\leq n^2\left(\frac{\max_{0\leq\theta\leq\alpha+o(1)} 4^{H(\theta)}\lambda^{2\theta}}{1+\lambda}\right)^n. \end{aligned}$$

We analyze this quantity later because it coincides with another quantity to appear. Let $Z_{\text{double}}(\lambda) = \sum_{I\in\mathcal{I}_{\mathcal{L}}\cap\mathcal{I}_{\mathcal{R}}} \lambda^{|I|}$. We also need to prove that $Z_{\text{double}}(\lambda)/Z(\lambda)$ is small because an independent $I \in \mathcal{I}_{\mathcal{L}} \cap \mathcal{I}_{\mathcal{R}}$ is counted twice in $Z_{\mathcal{L}}(\lambda) + Z_{\mathcal{R}}(\lambda)$. We have

$$\frac{Z_{\text{double}}(\lambda)}{Z(\lambda)} \le \frac{\left(\sum_{k=0}^{\lfloor \alpha n \rfloor} \binom{n}{k} \lambda^k\right)^2}{(1+\lambda)^n} \le \frac{\left(n \cdot \max_{0 \le \theta \le \alpha} 2^{H(\theta)n} \lambda^{\theta n}\right)^2}{(1+\lambda)^n}$$
$$\le n^2 \left(\frac{\max_{0 \le \theta \le \alpha + o(1)} 4^{H(\theta)} \lambda^{2\theta}}{1+\lambda} \right)^n.$$

It suffices to prove that $4^{H(\theta)}\lambda^{2\theta}/(1+\lambda) < 1$ for any $0 \le \theta \le \alpha + o(1)$. It is straightforward to verify that $\lambda^{2\theta}/(1+\lambda)$ is decreasing in λ on $[\lambda^*, +\infty)$ for any $0 \le \theta \le \alpha + o(1)$ (by taking the derivative of the log of this function). Therefore $4^{H(\theta)}\lambda^{2\theta}/(1+\lambda) \le 4^{H(\theta)}\lambda_*^{2\theta}/(1+\lambda_*)$. Taking log we obtain

$$2H(\theta) + 2\theta \log \lambda_* - \log(1+\lambda_*) \le 4(\alpha + o(1)) \log \frac{1}{\alpha + o(1)} - \lambda_*$$
$$\le 4(\alpha + o(1)) \log \Delta - \alpha (\log \Delta)^2 < 0$$

where we use Lemma 9 and $\log(1+x) \ge x$ for $x \in [0, 1]$. $\square$

3.2. Approximating $Z_{\mathcal{X}}(G, \lambda)$

In this part, we approximate $Z_{\mathcal{X}}(G, \lambda)$ for $\mathcal{X} \in \{\mathcal{L}, \mathcal{R}\}$ using the polymer model partition function. For any $I \in \mathcal{I}_{\mathcal{X}}(G)$, we can partition the graph $(G^2)[I \cap \mathcal{X}]$ (first obtain G^2 and then induce on $I \cap \mathcal{X}$) into maximal connected components $G_1, G_2, \dots, G_k$ for some $k \ge 0$ ($k = 0$ if $I \cap \mathcal{X} = \emptyset$). We shall think G_i as a (connected) subgraph of G^2. It is easy to verify that there are no edges in G^2 between G_i and G_j for any $i \ne j$. If $k = 0$, let polymers$(I) = \emptyset$; Otherwise let polymers$(I) = \left\{(G_i, \mathbf{1}_{V(G_i)}) : i \in [k]\right\}$ where $\mathbf{1}_{V(G_i)}$ is the unique mapping from $V(G_i)$ to $\{1\}$. Let

$$\Gamma_{\mathcal{X}}(G) = \bigcup_{I \in \mathcal{I}_{\mathcal{X}}(G)} \text{polymers}(I)$$

be the set of all polymers. Two polymers γ_1 and γ_2 are compatible if $d_{G^2}(\overline{\gamma_1}, \overline{\gamma_2}) > 1$, equivalent to $d_G(\overline{\gamma_1}, \overline{\gamma_2}) > 2$. We remark that an independent set I with $|I \cap \mathcal{X}| \le \alpha n$ is naturally decomposed to a compatible subset of polymers in $\Gamma_{\mathcal{X}}(G)$, which is polymers(I). For each polymer γ, define its weight function $w(\gamma, \cdot)$ as

$$w(\gamma, z) = \lambda^{|\overline{\gamma}|}(1+\lambda)^{-|N_G(\overline{\gamma})|} z^{|\overline{\gamma}|},$$

which can be computed in polynomial time in $\left|\overline{\gamma}\right|$. The partition function is

$$\Xi_{\mathcal{X}}(z) = \sum_{\Gamma \in \mathsf{cpt}(\Gamma_{\mathcal{X}}(G))} \prod_{\gamma \in \Gamma} w(\gamma, z).$$

Lemma 13. *For $\mathcal{X} \in \{\mathcal{L}, \mathcal{R}\}$ and $\lambda \in \mathbb{C}$,*

$$Z_{\mathcal{X}}(G, \lambda) = (1+\lambda)^n \sum_{\Gamma \in \mathsf{cpt}(\Gamma_{\mathcal{X}}(G)) : \left|\overline{\Gamma}\right| \le \alpha n} \prod_{\gamma \in \Gamma} w(\gamma, 1).$$

Proof. It follows from the definition of polymers that polymers$(I) \in \mathsf{cpt}(\Gamma_{\mathcal{X}})$ for each $I \in \mathcal{I}_{\mathcal{X}}$. Besides, for any $\Gamma \in \mathsf{cpt}(\Gamma_{\mathcal{X}})$ with $\left|\overline{\Gamma}\right| \le \alpha n$, $\overline{\Gamma}$ is the union of all polymers in polymers(I) if and only if $I \cap \mathcal{X} = V(\overline{\Gamma})$. Thus

$$Z_{\mathcal{X}}(G, \lambda) = \sum_{I \in \mathcal{I}_{\mathcal{X}}} \lambda^{|I|} = \sum_{\Gamma \in \mathsf{cpt}(\Gamma_{\mathcal{X}}) : \left|\overline{\Gamma}\right| \le \alpha n} \; \sum_{I \in \mathcal{I}_{\mathcal{X}} : I \cap \mathcal{X} = V(\overline{\Gamma})} \lambda^{|I|}$$
$$= \sum_{\Gamma \in \mathsf{cpt}(\Gamma_{\mathcal{X}}) : \left|\overline{\Gamma}\right| \le \alpha n} \lambda^{\left|\overline{\Gamma}\right|} (1+\lambda)^{\left|(\mathcal{L} \cup \mathcal{R}) \setminus (\mathcal{X} \cup N_G(\overline{\Gamma}))\right|}.$$

Since Γ is compatible, $N_G(\overline{\Gamma}) = \bigcup_{\gamma \in \Gamma} N_G(\overline{\gamma})$ and $\left|(\mathcal{L} \cup \mathcal{R}) \setminus (\mathcal{X} \cup N_G(\overline{\Gamma}))\right| = n - \sum_{\gamma \in \Gamma} \left|N_G(\overline{\gamma})\right|$. Then

$$Z_{\mathcal{X}}(G, \lambda) = \sum_{\Gamma \in \mathsf{cpt}(\Gamma_{\mathcal{X}}) : \left|\overline{\Gamma}\right| \le \alpha n} \lambda^{\sum_{\gamma \in \Gamma} |\overline{\gamma}|} (1+\lambda)^{n - \sum_{\gamma \in \Gamma} N_G(\overline{\gamma})}$$
$$= \sum_{\Gamma \in \mathsf{cpt}(\Gamma_{\mathcal{X}}) : \left|\overline{\Gamma}\right| \le \alpha n} (1+\lambda)^n \prod_{\gamma \in \Gamma} \lambda^{|\overline{\gamma}|} (1+\lambda)^{-|N_G(\overline{\gamma})|}$$
$$= (1+\lambda)^n \sum_{\Gamma \in \mathsf{cpt}(\Gamma_{\mathcal{X}}) : \left|\overline{\Gamma}\right| \le \alpha n} \prod_{\gamma \in \Gamma} w(\gamma, 1). \quad \square$$

Lemma 14. *For $\mathcal{X} \in \{\mathcal{L}, \mathcal{R}\}$, sufficiently large Δ and $\lambda \geq \lambda_*$,*

$$(1+\lambda)^n \Xi_{\mathcal{X}}(1) = (1+\lambda)^n \sum_{\Gamma \in \mathsf{cpt}(\Gamma_{\mathcal{X}}(G))} \prod_{\gamma \in \Gamma} w(\gamma, 1)$$

is a K^{-n}-relative approximation to $Z_{\mathcal{X}}(G,\lambda)$ for some constant $K(\Delta) > 1$.

Proof. Note that $(1+\lambda)^n \Xi_{\mathcal{X}}(1) \geq Z_{\mathcal{X}}(G,\lambda)$. Using $Z_{\mathcal{X}}(G,\lambda) \geq (1+\lambda)^n$, Lemma 13 and the expansion property we obtain

$$\epsilon = \frac{(1+\lambda)^n \Xi_{\mathcal{X}}(1) - Z_{\mathcal{X}}(G,\lambda)}{Z_{\mathcal{X}}(G,\lambda)} \leq \sum_{\Gamma \in \mathsf{cpt}(\Gamma_{\mathcal{X}})\colon |\overline{\Gamma}| > \alpha n} \prod_{\gamma \in \Gamma} w(\gamma, 1) \leq \sum_{\Gamma \in \mathsf{cpt}(\Gamma_{\mathcal{X}})\colon |\overline{\Gamma}| > \alpha n} \lambda^{|\overline{\Gamma}|} (1+\lambda)^{-\beta |\overline{\Gamma}|}.$$

Then we enumerate k vertices in $\mathcal{X}$ to represent Γ. This is loose but enough for our need.

$$\begin{aligned}
\epsilon &\leq \sum_{k=\lceil \alpha n \rceil}^{\lfloor n/\beta \rfloor} \binom{n}{k} \lambda^k (1+\lambda)^{-\beta k} \leq \sum_{k=\lceil \alpha n \rceil}^{\lfloor n/\beta \rfloor} 2^{H(k/n)n} \lambda^k (1+\lambda)^{-\beta k} \\
&\leq \sum_{k=\lceil \alpha n \rceil}^{\lfloor n/\beta \rfloor} \left(2^{H(k/n)n/k} \lambda (1+\lambda)^{-\beta} \right)^k \\
&\leq \sum_{k=\lceil \alpha n \rceil}^{\lfloor n/\beta \rfloor} \left(4^{\log(n/k)} \lambda (1+\lambda)^{-\beta} \right)^k \\
&\leq \sum_{k=\lceil \alpha n \rceil}^{\lfloor n/\beta \rfloor} \left((1+\lambda)^{1-\beta} / \alpha^2 \right)^k,
\end{aligned}$$

where we use Lemma 8 and Lemma 9. It remains to prove that $(1+\lambda)^{1-\beta}/\alpha^2 < 1$. Taking log we obtain

$$\begin{aligned}
\log\left((1+\lambda)^{1-\beta}/\alpha^2 \right) &\leq 2 \log \frac{1}{\alpha} - (\beta - 1) \log(1+\lambda_*) \\
&\leq 2 \log \frac{1}{\alpha} - (\beta - 1) \lambda_* \\
&\leq 2 \log \Delta - (1 - 2\alpha) (\log \Delta)^2 < 0,
\end{aligned}$$

where we use the inequality $\log(1+x) \geq x$ for $x \in [0,1]$. □

3.3. Approximating the polymer model partition function

Lemma 15. *For sufficiently large Δ, there is an FPTAS for $\Xi_{\mathcal{X}}(1)$ for all $G = (\mathcal{L}, \mathcal{R}, E) \in \mathcal{G}^{\Delta}_{\alpha,\beta}$, $\mathcal{X} \in \{\mathcal{L}, \mathcal{R}\}$ and $\lambda \geq \lambda_*$.*

Proof. We use the FPTAS in Theorem 5 to design the FPTAS we need. To this end, we generate a graph $H = G^2[\mathcal{X}]$ (first obtain G^2 and then induce on $\mathcal{X}$) in polynomial time in $|G|$ for $G \in \mathcal{G}^{\Delta}_{\alpha,\beta}$. Let H be the input to the FPTAS in Theorem 5 and we also allow the FPTAS to query the original graph G. Note that $|V(H)| = |\mathcal{L}| = |\mathcal{R}|$. Let $\Gamma(H) = \Gamma_{\mathcal{X}}(G)$ and $\Xi(H, z) = \Xi_{\mathcal{X}}(z) = \sum_{\Gamma \in \mathsf{cpt}(\Gamma_{\mathcal{X}}(G))} \prod_{\gamma \in \Gamma} w(\gamma, z)$ where $w(\gamma, z) = \lambda^{|\overline{\gamma}|} (1+\lambda)^{-|N_G(\overline{\gamma})|} z^{|\overline{\gamma}|}$. Fix $\lambda \geq \lambda_*$ and we verify that the conditions in Theorem 5 hold:

- The degree of a monomial $\prod_{\gamma \in \Gamma} w(\gamma, z)$ is at most $|V(H)|$ since Γ is compatible.
- The coefficient $\lambda^{|\overline{\gamma}|}(1+\lambda)^{-|N_G(\overline{\gamma})|}$ can be computed by enumerating each vertex in $V(\overline{\gamma})$ and its adjacent vertices in G. This step takes $O\left(|V(\overline{\gamma})| \cdot |V(G)|\right) \leq \exp\left(O\left(|\overline{\gamma}| + \log |H|\right)\right)$ time.
- For every connected subgraph H' of H, there is exactly one polymer in $\Gamma(H)$ such that $\overline{\gamma} = H'$.
- This one follows from Lemma 16. □

Lemma 16. *There is a constant $R > 1$ so that $\Xi_{\mathcal{X}}(z) \neq 0$ for sufficiently large Δ, $\lambda \geq \lambda_*$, $G = (\mathcal{L}, \mathcal{R}, E) \in \mathcal{G}^{\Delta}_{\alpha,\beta}$, $\mathcal{X} \in \{\mathcal{L}, \mathcal{R}\}$ and $z \in \mathbb{C}$ with $|z| < R$.*

Proof. Let $R = 2$. For any polymer γ, let $a(\gamma) = |\overline{\gamma}|$. We will verify that the KP-condition

$$\sum_{\gamma' : \gamma' \nsim \gamma} e^{|\overline{\gamma'}|} |w(\gamma', z)| \leq |\overline{\gamma}|$$

holds for any γ and $|z| < R$. It then follows from Lemma 6 that $\Xi_{\mathcal{X}}(z) \neq 0$ for any $|z| < R$. Note that if $\gamma' \nsim \gamma$ then $d_{G^2}(\overline{\gamma'}, \overline{\gamma}) \leq 1$. Thus

$$\sum_{\gamma':\gamma' \nsim \gamma} e^{\left|\overline{\gamma'}\right|}\left|w(\gamma', z)\right| = \sum_{\gamma':\gamma \nsim \gamma} e^{\left|\overline{\gamma'}\right|}\left|w(\gamma', 1)\right| \cdot |z|^{\left|\overline{\gamma'}\right|} \leq \Delta^2 \left|\overline{\gamma}\right| \sum_{k=1}^{\lfloor \alpha n \rfloor} (e\Delta^2)^{k-1} e^k (1+\lambda)^{-\beta k} R^k$$
$$\leq \left|\overline{\gamma}\right| \sum_{k=1}^{\infty} \left(e^2 \Delta^2 (1+\lambda)^{-\beta} R\right)^k .$$

It is sufficient to verify that $e^2\Delta^2(1+\lambda)^{-\beta} R \leq \frac{1}{2}$ for all $\lambda \geq \lambda_*$. Taking log we obtain

$$\begin{aligned} 2\log e + 2\log\Delta + 1 - \beta\log(1+\lambda) &\leq 2\log e + 2\log\Delta + 1 - \beta\lambda_* \\ &\leq 2\log e + 2\log\Delta + 1 - (1-\alpha)(\log\Delta)^2 \\ &\leq -1, \end{aligned}$$

where we use the inequality $\log(1+x) \geq x$ for $x \in [0, 1]$. □

3.4. Putting things together

Combining previous results, we obtain our main result for counting independent sets below, which simply follows from Lemma 11 and Lemma 17. In fact, the theorem here is a little bit stronger since $\lambda_* < 4(\log\Delta)^3/\Delta$.

Theorem 1. *For sufficiently large Δ, there is an FPTAS such that with high probability (tending to 1 as $n \to \infty$) for a graph G chosen uniformly at random from $\mathcal{G}^{\mathrm{bip}}_{n,\Delta}$ the FPTAS computes $Z(G, \lambda)$ for $\lambda \geq \lambda_*$.*

Algorithm 1 Counting independent sets for sufficiently large Δ and $\lambda \geq \lambda_*$.

```
Input: A graph G = (L, R, E) ∈ G^Δ_{α,β} with n vertices on both partitions and ε > 0
Output: Ẑ such that exp(−ε)Ẑ ≤ Z(G, λ) ≤ exp(ε)Ẑ
if n ≤ N or ε ≤ 2K^{−n} then
    Use the brute-force algorithm to compute Ẑ ← Z(G, λ);
    Exit;
end if
ε' ← ε − K^{−n};
Use the FPTAS in Lemma 15 to obtain Ẑ_L and Ẑ_R, which are ε'-relative approximations to Ξ_L(1) and Ξ_R(1), respectively.
Ẑ ← (1 + λ)^n (Ẑ_L + Ẑ_R);
```

Lemma 17. *For sufficiently large Δ, there is an FPTAS for $Z(G, \lambda)$ for all $G \in \mathcal{G}^{\Delta}_{\alpha,\beta}$ and $\lambda \geq \lambda_*$.*

Proof. First we state our algorithm. See Algorithm 1 for a pseudocode description. The input is a graph $G = (\mathcal{L}, \mathcal{R}, E) \in \mathcal{G}^{\Delta}_{\alpha,\beta}$ and an approximation parameter $\varepsilon > 0$. The output is a number $\widehat{Z}$ to approximate $Z(G, \lambda)$. We use $\Xi_{\mathcal{X}}(z)$ to denote the partition function of the polymer model $\Gamma_{\mathcal{X}}(G)$ for $\mathcal{X} \in \{\mathcal{L}, \mathcal{R}\}$. Let K_1, K_2 be the constants in Lemma 12 and Lemma 14, respectively. These two lemmas show that, since $n > N$ is sufficiently large, $(1+\lambda)^n (\Xi_{\mathcal{L}}(1) + \Xi_{\mathcal{R}}(1))$ is a $K_1^{-n} + K_2^{-n} \leq 2\min(K_1, K_2)^{-n} \leq K^{-n}$-relative approximation to $Z(G, \lambda)$ for some constant $K > 1$. If $n \leq N$ or $\varepsilon \leq 2K^{-n}$, we use the brute-force algorithm to compute $Z(G, \lambda)$. If $\varepsilon > 2K^{-n}$, we apply the FPTAS in Lemma 15 with approximation parameter $\varepsilon' = \varepsilon - K^{-n}$ to obtain $\widehat{Z}_{\mathcal{L}}$ and $\widehat{Z}_{\mathcal{R}}$ which approximate $\Xi_{\mathcal{L}}(1)$ and $\Xi_{\mathcal{R}}(1)$, respectively. Let $\widehat{Z} = (1+\lambda)^n(\widehat{Z}_{\mathcal{L}} + \widehat{Z}_{\mathcal{R}})$ be the output. It is clear that $\exp(-\varepsilon)\widehat{Z} \leq Z(G, \lambda) \leq \exp(\varepsilon)\widehat{Z}$.

Then we show that Algorithm 1 is indeed an FPTAS. It is required that the running time of our algorithm is bounded by $(n/\varepsilon)^{O(1)}$ for all large n. Suppose $n > N$. If $\varepsilon \leq 2K^{-n}$, the running time of the algorithm would be $2.1^n \leq (n/\varepsilon)^{O(1)}$; Otherwise the running time of the algorithm would be $\left(n/\varepsilon'\right)^{O(1)} = \left(n/(\varepsilon - K^{-n})\right)^{O(1)} \leq (2n/\varepsilon)^{O(1)} = (n/\varepsilon)^{O(1)}$. □

4. A special case

In this section, we prove Theorem 2. To obtain this result, we need to set different values for parameters α, β, though still depending on Δ, and it is sufficient to obtain Theorem 2 by establishing counterparts of Lemmas 11, 12, 14 and 16.

Let $\phi = 0.773$, $\psi = 0.273$, $\alpha = \frac{\phi}{\psi\Delta}$, $\beta = \psi\Delta = \frac{\phi}{\alpha}$. Other definitions and notations are the same as previous section. We abbreviate notations like $Z(G, 1)$ to $Z(G)$.

Lemma 18. *For integers $\Delta \geq 50$, $\lim_{n\to\infty} \Pr_{G\sim\mathcal{G}^{\text{bip}}_{n,\Delta}} \left[G \in \mathcal{G}^{\Delta}_{\alpha,\beta}\right] = 1$.*

Proof. We verify the condition in Lemma 4. Apply Lemma 10 ($y = \frac{1}{\phi} > 1, x = \alpha < 1/y = \phi$),

$$H(\alpha) - \alpha\beta H(1/\beta) = H(\alpha) - \phi H(\alpha/\phi) \geq \alpha\log(1/\phi).$$

Then

$$R(\Delta) = \frac{H(\alpha)+H(\alpha\beta)}{H(\alpha)-\alpha\beta H(1/\beta)} \cdot \frac{1}{\Delta} \leq \frac{H(\alpha)+H(\phi)}{\alpha\log(1/\phi)} \cdot \frac{\psi\alpha}{\phi} = \frac{\psi(H(\alpha)+H(\phi))}{\phi\log(1/\phi)}$$

where we use Lemma 9. For $\Delta \geq 59$, it is clear that the bound we obtained for $R(\Delta)$ is decreasing as Δ increasing. Thus

$$R(\Delta) \leq \left.\frac{\psi(H(\alpha)+H(\phi))}{\phi\log(1/\phi)}\right|_{\Delta=59} \approx 0.998 < 1.$$

For $\Delta \in [49, 58]$, we can directly compute the value of R:

Δ	49	50	51	52	53	54	55	56	57	58
R(Δ)	1.003	0.9994	0.996	0.992	0.989	0.986	0.982	0.98	0.977	0.974

□

Lemma 19. *For $\Delta \geq 50$, $Z_{\mathcal{L}}(G) + Z_{\mathcal{R}}(G)$ is a K^{-n}-relative approximation to $Z(G)$ for some constant $K(\Delta) > 1$.*

Proof. As in the proof of Lemma 12, let $Z_{\text{omitted}} = |\mathcal{I} \setminus (\mathcal{I}_{\mathcal{L}} \cup \mathcal{I}_{\mathcal{R}})|$ and $Z_{\text{double}} = |\mathcal{I}_{\mathcal{L}} \cap \mathcal{I}_{\mathcal{R}}|$. Then

$$\begin{aligned}\frac{Z_{\text{omitted}}}{Z} &\leq \frac{\sum_{k=\lceil\alpha n\rceil}^{\lfloor n-\lfloor\alpha n\rfloor\beta\rfloor}\binom{n}{k}2^{n-\beta\lfloor\alpha n\rfloor}}{2^n} \leq \frac{n\cdot 2^{H(1-(\alpha n-1)\beta/n)n}2^{n-\beta(\alpha n-1)}}{2^n}\\ &= n\cdot\left(2^{H(1-\phi+\beta/n)+(1-\phi+\beta/n)-1}\right)^n.\end{aligned}$$

We verify that (taking $\beta/n = 10^{-5}$) $H(1-\phi+\beta/n) + (1-\phi+\beta/n) \approx 0.9998 < 1$. Besides,

$$\frac{Z_{\text{double}}}{Z} \leq \frac{\left(\sum_{k=0}^{\lfloor\alpha n\rfloor}\binom{n}{i}\right)^2}{2^n} \leq n^2\left(2^{2H(\alpha)-1}\right)^n.$$

It holds that $2H(\alpha) \leq 2H(\alpha)|_{\Delta=50} \approx 0.63 < 1$. □

Lemma 20. *For $\mathcal{X} \in \{\mathcal{L}, \mathcal{R}\}$ and $\Delta \geq 50$,*

$$2^n\Xi_{\mathcal{X}}(1) = 2^n\sum_{\Gamma\in\mathsf{cpt}(\Gamma_{\mathcal{X}}(G))}\prod_{\gamma\in\Gamma}w(\gamma,1)$$

is a K^{-n}-relative approximation to $Z_{\mathcal{X}}(G)$ for some constant $K(\Delta) > 1$.

Proof. We use the same proof strategy as in the proof of Lemma 14. Then we simply need to verify that

$$\alpha^2\cdot 2^\beta \geq \alpha^2\cdot 2^\beta|_{\Delta=50} \approx 41 > 1. \quad \square$$

Lemma 21. *There is a constant $R > 1$ so that $\Xi_{\mathcal{X}}(z) \neq 0$ for $\Delta \geq 50$, $G = (\mathcal{L}, \mathcal{R}, E) \in \mathcal{G}^{\Delta}_{\alpha,\beta}$, $\mathcal{X} \in \{\mathcal{L}, \mathcal{R}\}$ and $z \in \mathbb{C}$ with $|z| < R$.*

Proof. Let $R = 1 + 10^{-4}$. For any polymer γ, let $a(\gamma) = t\cdot|\overline{\gamma}|$ where $t = 0.25$. We verify the KP-condition

$$\sum_{\gamma':\gamma'\nsim\gamma} e^{a(\gamma)}\left|w(\gamma',z)\right| \leq a(\gamma)$$

holds for any γ and $|z| < R$. It then follows from Lemma 6 that $\Xi_{\mathcal{X}}(z) \neq 0$ for any $|z| < R$. Note that for a vertex $v \in G^2[\mathcal{X}]$ (first obtain G^2), its degree is at most $d = \Delta(\Delta-1)$. Therefore,

$$\sum_{\gamma':\gamma'\nsim\gamma} e^{t|\gamma|}\left|w(\gamma,z)\right| \leq (d+1)|\overline{\gamma}|\left(e^t2^{-\Delta}R + \sum_{k=2}^{\lfloor\alpha n\rfloor}(ed)^{k-1}2^{-1}e^{tk}2^{-\beta k}R^k\right)$$

where we use Lemma 7 to count the number of incompatible polymers of a fixed size. Let $x = e^{t+1}d2^{-\beta}R$. Then

$$\sum_{\gamma':\gamma'\not\sim\gamma} e^{t|\gamma|}\left|w(\gamma,z)\right| \le \frac{x}{e}\cdot\frac{d+1}{d}\cdot\left|\overline{\gamma}\right|\left(2^{-(\Delta-\beta)}+\frac{1}{2}\sum_{k=2}^{\infty}x^{k-1}\right)$$
$$= \frac{x(d+1)}{ed}\left|\overline{\gamma}\right|\left(2^{-(\Delta-\beta)}+\frac{x}{2(1-x)}\right). \tag{1}$$

It is straightforward to verify that $d2^{-\beta}$ is decreasing in Δ on $[50,+\infty)$. Thus

$$\text{Equation (1)} \le \frac{x(d+1)}{ed}\left|\overline{\gamma}\right|\left(2^{-(\Delta-\beta)}+\frac{x}{2(1-x)}\right)\Big|_{\Delta=50} \approx 0.243\left|\overline{\gamma}\right| < t\left|\overline{\gamma}\right| = 0.25\left|\overline{\gamma}\right|. \quad \square$$

5. Counting colorings

Throughout this section, we consider $q \ge 3$, $\Delta \ge 80q^3\log^2 q$. Set parameters $\alpha,\beta > 0$ such that

$$\Delta = \frac{-4\log\alpha}{\alpha},$$
$$\beta = 1/\alpha - 1.$$

It is easy to verify that $1/\alpha > 4q^3\log q$. First we show that almost all Δ-regular bipartite graphs are (α,β)-expanders.

Lemma 22. *For $q \ge 3$ and $\Delta \ge 80q^3\log^2 q$, $\lim_{n\to\infty}\Pr_{G\sim\mathcal{G}^{\mathrm{bip}}_{n,\Delta}}\left[G\in\mathcal{G}^{\Delta}_{\alpha,\beta}\right]=1$.*

Proof. We only need to verify the condition of Lemma 4. By Lemma 10 ($y = \frac{1}{\alpha\beta} \ge 1, x = \alpha \le 1/y$),

$$H(\alpha) - \alpha\beta H(1/\beta) \ge \alpha\log(\alpha\beta)^{-1} = -\alpha\log(1-\alpha) > \alpha^2.$$

Therefore, we have

$$\frac{H(\alpha)+H(\alpha\beta)}{H(\alpha)-\alpha\beta H(1/\beta)} < \frac{2H(\alpha)}{\alpha^2} \le \frac{-4\alpha\log\alpha}{\alpha^2} = \Delta,$$

where the second inequality follows from Lemma 9. $\square$

In the rest of this section, let $G = (\mathcal{L},\mathcal{R},E) \in \mathcal{G}^{\Delta}_{\alpha,\beta}$ be a bipartite graph with n vertices on both partitions. Moreover, we assume that $n > N$ for some sufficiently large constant $N = N(q) > 0$.

Recall that we use $\mathcal{C}(G)$ to denote the set of all proper colorings of G. For a nonempty set $X \subsetneq [q]$ of colors, it naturally induces a "ground cluster" of colorings τ such that τ only assigns colors from the set X (resp. $[q]\setminus X$) to the vertex set $\mathcal{L}$ (resp. $\mathcal{R}$). Note that τ is always a proper coloring and the number of such colorings is $|X|^n(q-|X|)^n$. Similar to the case of independent set, we only need to consider the colorings that are "close" to ground clusters. It is even better here that any proper coloring is "close" to some ground cluster. Here the "distance" between a coloring σ and the ground cluster induced by X is defined as $d_X(\sigma) = \left|\sigma_{\mathcal{L}}^{-1}([q]\setminus X)\right| + \left|\sigma_{\mathcal{R}}^{-1}(X)\right|$, counting the number of vertices assigned "wrong" colors. We define

$$\mathcal{C}_X(G) = \{\sigma\in\mathcal{C}(G) : d_X(\sigma) < \delta n\}$$

where $\delta = q(\alpha+\beta/N) \approx q\alpha$. For convenience, let $\mathcal{C}_{\emptyset} = \mathcal{C}_{[q]} = \emptyset$.

Lemma 23. *For any proper coloring $\sigma\in\mathcal{C}(G)$, there exists some $X \subseteq [q]$ such that $\sigma\in\mathcal{C}_X(G)$.*

Proof. We define

$$X = \left\{c\in[q] : \left|\sigma_{\mathcal{L}}^{-1}(c)\right| \ge \alpha n\right\}.$$

Because $q\alpha < 1$, the set X is nonempty. For any $c \in X$, since G is an (α,β)-expander,

$$\left|\sigma_{\mathcal{R}}^{-1}(c)\right| \le n - \left|N(\sigma_{\mathcal{L}}^{-1}(c))\right| \le n - \beta\lfloor\alpha n\rfloor \le \alpha n + \beta.$$

Thus $X \ne [q]$ because $q(\alpha n+\beta) < \delta n < n$. Moreover, $\left|\sigma_{\mathcal{L}}^{-1}(c)\right| < \alpha n$ for any color $c \notin X$. Now we can simply bound

$$\begin{aligned} d_X(\sigma) &= \left|\sigma_{\mathcal{L}}^{-1}([q]\setminus X)\right| + \left|\sigma_{\mathcal{R}}^{-1}(X)\right| \\ &< (q-|X|)\alpha n + |X|(\alpha n+\beta) \\ &< \delta n. \end{aligned}$$

So $\sigma \in \mathcal{C}_X(G)$. □

5.1. Approximating $|\mathcal{C}(G)|$

In this subsection, we simply write $\mathcal{C} = \mathcal{C}(G)$ and $\mathcal{C}_X = \mathcal{C}_X(G)$. The main result is that we can use $\sum_{X:\,|X|\in\left\{\underline{q},\overline{q}\right\}} |\mathcal{C}_X|$ to approximate $|\mathcal{C}|$ (recall that $\underline{q} = \lfloor q/2\rfloor, \overline{q} = \lceil q/2\rceil$). The following lemma follows from Lemma 25 and Lemma 26.

Lemma 24. *Let $Z = \sum_{X\subseteq[q]:|X|\in\left\{\underline{q},\overline{q}\right\}} |\mathcal{C}_X|$. Then Z is a K^{-n}-relative approximation to $|\mathcal{C}|$ for some constant $K(q) > 1$.*

Lemma 25. *$\sum_{X\subseteq[q]} |\mathcal{C}_X|$ is a K^{-n}-relative approximation to $|\mathcal{C}|$ for some constant $K(q) > 1$.*

Proof. Fix two sets $\emptyset \subsetneq X \neq Y \subsetneq [q]$. For any $\sigma \in \mathcal{C}_X \cap \mathcal{C}_Y$, most of the vertices on the LHS are in colors $X\cap Y$ and most of the RHS in $[q]\setminus(X\cup Y)$. Formally, we have

$$\begin{aligned} \left|\sigma_{\mathcal{L}}^{-1}([q]\setminus(X\cap Y))\right| + \left|\sigma_{\mathcal{R}}^{-1}(X\cup Y)\right| &\le \left(\left|\sigma_{\mathcal{L}}^{-1}([q]\setminus X)\right| + \left|\sigma_{\mathcal{L}}^{-1}([q]\setminus Y)\right|\right) + \left(\left|\sigma_{\mathcal{R}}^{-1}(X)\right| + \left|\sigma_{\mathcal{R}}^{-1}(Y)\right|\right) \\ &= \left(\left|\sigma_{\mathcal{L}}^{-1}([q]\setminus X)\right| + \left|\sigma_{\mathcal{R}}^{-1}(X)\right|\right) + \left(\left|\sigma_{\mathcal{L}}^{-1}([q]\setminus Y)\right| + \left|\sigma_{\mathcal{R}}^{-1}(Y)\right|\right) \\ &< 2\delta n. \end{aligned}$$

Thus we can upper bound

$$|\mathcal{C}_X\cap\mathcal{C}_Y| \le \binom{2n}{\lfloor 2\delta n\rfloor} q^{\lfloor 2\delta n\rfloor} |X\cap Y|^n |[q]\setminus(X\cup Y)|^n \le \left(4^{H(\delta)} q^{2\delta} \underline{q}(\overline{q}-1)\right)^n,$$

where the second inequality follows from Lemma 8 and $|X\cap Y| + |[q]\setminus(X\cup Y)| \le q-1$. It is clear that $|\mathcal{C}| \ge \underline{q}^n \overline{q}^n$ and we obtain

$$\frac{|\mathcal{C}_X\cap\mathcal{C}_Y|}{|\mathcal{C}|} \le \left(4^{H(\delta)} q^{2\delta}(1-1/\overline{q})\right)^n.$$

Set $K' = 4^{H(\delta)}q^{2\delta}(1-1/\overline{q})$ and $K = 1/K'$. Recall that $|\mathcal{C}| = \left|\bigcup_{X\subseteq[q]}\mathcal{C}_X\right|$, so we have

$$|\mathcal{C}| \le \sum_{X\subseteq[q]} |\mathcal{C}_X| \le |\mathcal{C}| + \sum_{X\neq Y} |\mathcal{C}_X\cap\mathcal{C}_Y| \le (1+4^q K'^n)|\mathcal{C}|.$$

It remains to show that $K' < 1$:

$$\begin{aligned} \log K' &= 2H(\delta) + 2\delta\log q + \log(1-1/\overline{q}) \\ &< -2q\alpha\log q\alpha^2 - \frac{\log e}{\overline{q}} \\ &< 2q\frac{1}{4q^3\log q}\log q(4q^3\log q)^2 - 2/q < 0. \end{aligned}$$

Here we use the fact that $1/\alpha > 4q^3\log q$. □

Lemma 26. *Let $Z = \sum_{X\subseteq[q]:|X|\in\left\{\underline{q},\overline{q}\right\}} |\mathcal{C}_X|$. Then Z is a K^{-n}-relative approximation to $\sum_{X\subseteq[q]}|\mathcal{C}_X|$ for some constant $K(q) > 1$.*

Proof. Let Y be any subset of $[q]$ such that $|Y| < \underline{q}$ or $|Y| > \overline{q}$. Then we have

$$|\mathcal{C}_Y| \le \binom{2n}{\lfloor\delta n\rfloor} q^{\lfloor\delta n\rfloor} |Y|^n (q-|Y|)^n \le \left(4^{H(\delta/2)} q^{\delta}(\underline{q}-1)(\overline{q}+1)\right)^n,$$

where the second inequality follows from Lemma 8 and $|Y|(q-|Y|) \le (\underline{q}-1)(\overline{q}+1)$. Clearly $Z \ge \underline{q}^n\overline{q}^n$ and we obtain

$$\frac{|\mathcal{C}_Y|}{Z} \le \left(4^{H(\delta/2)} q^{\delta} (1-1/\underline{q})(1+1/\overline{q})\right)^n \le \left(4^{H(\delta/2)} q^{\delta} (1-1/\overline{q}^2)\right)^n.$$

Set $K' = 4^{H(\delta/2)} q^{\delta}(1-1/\overline{q}^2)$ and $K = 1/K'$. Then

$$Z \le \sum_{X \subseteq [q]} |\mathcal{C}_X| = Z + \sum_{Y : |Y| \notin \{\underline{q}, \overline{q}\}} |\mathcal{C}_Y| \le (1 + 2^q K'^n) Z.$$

Now we show that $K' < 1$:

$$\begin{aligned} \log K' &= 2H(\delta/2) + \delta \log q + \log(1 - 1/\overline{q}^2) \\ &< -q\alpha \log(q\alpha^2/4) - 3/q^2 \\ &< \frac{\log(64 q^5 \log^2 q)}{4q^2 \log q} - 3/q^2 < 0. \end{aligned}$$

This completes the proof. $\square$

5.2. Approximating $|\mathcal{C}_X(G)|$

In this subsection, we use the polymer model partition function to approximate $|\mathcal{C}_X(G)|$ for $|X| \in \{\underline{q}, \overline{q}\}$. For any $\sigma \in \mathcal{C}_X(G)$, let $U = \{v \in \mathcal{L} : \sigma(v) \in [q] \setminus X\} \cup \{v \in \mathcal{R} : \sigma(v) \in X\}$. We can partition the graph $(G^2)[U]$ (first obtain G^2 and then induce on U) into maximal connected components $G_1, G_2, \ldots, G_k$ for some $k \ge 0$ ($k = 0$ if $U = \emptyset$). We shall think G_i as a (connected) subgraph of G^2. There are no edges in G^2 between G_i and G_j for any $i \ne j$. If $k = 0$, let polymers$(\sigma) = \emptyset$. For $k > 0$, let polymers$(\sigma) = \left\{ \left(G_i, \sigma_{V(G_i)}\right) : i \in [k] \right\}$ (recall that $\sigma_{V(G_i)}$ is σ restricting to the set $V(G_i)$). Let

$$\Gamma_X(G) = \bigcup_{\sigma \in \mathcal{C}_X(G)} \text{polymers}(\sigma),$$

be the set of all polymers. Two polymers γ_1, γ_2 are compatible if $d_{G^2}(\overline{\gamma_1}, \overline{\gamma_2}) > 1$, equivalent to $d_G(\overline{\gamma_1}, \overline{\gamma_2}) > 2$. For each polymer $\gamma \in \Gamma_X$, define its weight function $w(\gamma, \cdot)$ as

$$w(\gamma, z) = \frac{\left|\mathcal{D}_\gamma\right|}{|X|^n (q - |X|)^n} z^{|\overline{\gamma}|},$$

where $\mathcal{D}_\gamma$ is the set of colorings $\sigma \in \mathcal{C}_X(G)$ such that polymers$(\sigma) = \{\gamma\}$. The size of $\mathcal{D}_\gamma$ can be computed in polynomial time in $|\overline{\gamma}|$. In fact,

$$\left|\mathcal{D}_\gamma\right| = \left(\prod_{v \in \mathcal{L} \setminus V(\overline{\gamma})} |X \setminus Q_v| \right) \left(\prod_{v \in \mathcal{R} \setminus V(\overline{\gamma})} |([q] \setminus X) \setminus Q_v| \right),$$

where $Q_v = \omega_{\overline{\gamma}}(N(v) \cap V(\overline{\gamma}))$ is the set of colors for v that are occupied by its neighbors. The partition function is

$$\Xi_X(z) = \sum_{\Gamma \in \mathsf{cpt}(\Gamma_X(G))} \prod_{\gamma \in \Gamma} w(\gamma, z).$$

For $\Gamma \in \mathsf{cpt}(\Gamma_X)$, let $\mathcal{D}_\Gamma$ be the set of colorings $\sigma \in \mathcal{C}_X(G)$ such that polymers$(\sigma) = \Gamma$.

Lemma 27. $|\mathcal{C}_X(G)| = \sum_{\Gamma \in \mathsf{cpt}(\Gamma_X) : \left|\overline{\Gamma}\right| < \delta n} |\mathcal{D}_\Gamma|.$

Proof. It is sufficient to show that the set

$$\left\{ \mathcal{D}_\Gamma : \Gamma \in \mathsf{cpt}(\Gamma_X) \wedge \left|\overline{\Gamma}\right| < \delta n \right\}$$

is a partition of $\mathcal{C}_X$. By definition, $\mathcal{D}_\Gamma \subseteq \mathcal{C}_X$, and $\mathcal{D}_{\Gamma_1} \cap \mathcal{D}_{\Gamma_2} = \emptyset$ if $\Gamma_1 \ne \Gamma_2$. For any $\sigma \in \mathcal{C}_X$, the set $\Gamma = \text{polymers}(\sigma)$ is compatible and $\left|\overline{\Gamma}\right| < \delta n$, so $\sigma \in \mathcal{D}_\Gamma$. $\square$

Lemma 28. *For* $\Gamma \in \mathsf{cpt}(\Gamma_X)$, $|\mathcal{D}_\Gamma| = |X|^n (q - |X|)^n \prod_{\gamma \in \Gamma} w(\gamma, z).$

Proof. The conclusion is trivial if $|\Gamma| = 1$. Suppose that $|\Gamma| > 1$. Let $\Gamma = \Gamma_1 \cup \gamma$. Since $d_G(\overline{\Gamma_1}, \overline{\gamma}) > 2$, we have

$$|\mathcal{D}_\Gamma| = \left|\mathcal{D}_{\Gamma_1}\right| \cdot \frac{\left|\mathcal{D}_\gamma\right|}{|X|^n (q - |X|)^n} = w(\gamma, 1)\left|\mathcal{D}_{\Gamma_1}\right|.$$

And by induction, $\left|\mathcal{D}_{\Gamma_1}\right| = |X|^n (q - |X|)^n \prod_{\gamma' \in \Gamma_1} w(\gamma', 1)$. $\square$

With Lemmas 27 and 28, we are able to write $|\mathcal{C}_X(G)|$ as the sum of product of polymer weights with certain restriction, which can be approximated by the partition function of the polymer model.

Lemma 29. *There exists some constant $K = K(q) > 1$ such that*

$$|X|^n (q - |X|)^n \Xi_X(1) = |X|^n (q - |X|)^n \sum_{\Gamma \in \mathsf{cpt}(\Gamma_X)} \prod_{\gamma \in \Gamma} w(\gamma, 1)$$

is a K^{-n}-relative approximation to $|\mathcal{C}_X(G)|$ for $|X| \in \left\{\underline{q}, \overline{q}\right\}$.

The approximation works because the underlying graph G has strong expansion such that $\left|\overline{\Gamma}\right|$ cannot be too large and the weight $\prod_{\gamma \in \Gamma} w(\gamma, 1)$ decays exponentially as the size grows.

Lemma 30. *Let $\theta = \beta \lfloor \alpha n \rfloor / \lfloor \delta n \rfloor \approx \beta / q$ and $|X| \in \left\{\underline{q}, \overline{q}\right\}$. Then for any $\gamma \in \Gamma_X$,*

$$w(\gamma, 1) \le \left(1 - 1/\overline{q}\right)^{\theta |\overline{\gamma}|}.$$

For $\Gamma \in \mathsf{cpt}(\Gamma_X)$, it holds that $\left|\overline{\Gamma}\right| < 2n/\theta$ and

$$\prod_{\gamma \in \Gamma} w(\gamma, 1) \le \left(1 - 1/\overline{q}\right)^{\theta \left|\overline{\Gamma}\right|}.$$

Proof. For $\gamma \in \Gamma_X$, there are two cases:

- $\left|\overline{\gamma}\right| \le \alpha n$. Since the graph G is an (α, β)-expander, we have $\left|\overline{\gamma}\right| + \left|N(\overline{\gamma})\right| \ge \beta \left|\overline{\gamma}\right|$. In particular, if $q = 3$, then $\overline{\gamma} \subseteq \mathcal{L}$ or $\overline{\gamma} \subseteq \mathcal{R}$, and hence $\left|N(\overline{\gamma})\right| \ge \beta \left|\overline{\gamma}\right|$.
- $\alpha n < \left|\overline{\gamma}\right| < \delta n$. In this case, $\left|\overline{\gamma}\right| + \left|N(\overline{\gamma})\right| \ge \beta \lfloor \alpha n \rfloor \ge \theta \left|\overline{\gamma}\right|$.

Therefore, it holds that

$$w(\gamma, 1) = \frac{\left|\mathcal{D}_\gamma\right|}{|X|^n (q - |X|)^n} \le (1 - 1/\overline{q})^{|N(\overline{\gamma})|} \underline{q}^{-|\overline{\gamma}|} \le (1 - 1/\overline{q})^{\theta |\overline{\gamma}|}.$$

Let $\Gamma \subseteq \Gamma_X$ be compatible. Then $\left|\overline{\Gamma}\right| = \sum_{\gamma \in \Gamma} \left|\overline{\gamma}\right|$, and

$$\prod_{\gamma \in \Gamma} w(\gamma, 1) \le \prod_{\gamma \in \Gamma} (1 - 1/\overline{q})^{\theta |\overline{\gamma}|} = (1 - 1/\overline{q})^{\theta |\overline{\Gamma}|}.$$

For two different polymers $\gamma_1, \gamma_2 \in \Gamma$, $d_G(\overline{\gamma}_1, \overline{\gamma}_2) > 2$. Thus

$$2n \ge \sum_{\gamma \in \Gamma} \left|\overline{\gamma}\right| + \left|N(\overline{\gamma})\right| \ge \sum_{\gamma \in \Gamma} \theta \left|\overline{\gamma}\right| = \theta \left|\overline{\Gamma}\right|.$$

That is, $\left|\overline{\Gamma}\right| \le 2n/\theta$. $\square$

Proof of Lemma 29. Clearly $|\mathcal{C}_X| \ge |X|^n (q - |X|)^n$. Then

$$\begin{aligned}
\epsilon = \frac{|X|^n (q - |X|)^n \Xi_X(1) - |\mathcal{C}_X|}{|\mathcal{C}_X|} &\le \sum_{\Gamma \in \mathsf{cpt}(\Gamma_X) :\, \left|\overline{\Gamma}\right| \ge \delta n} \prod_{\gamma \in \Gamma} w(\gamma, 1) \\
&\le \sum_{\Gamma \in \mathsf{cpt}(\Gamma_X) :\, \left|\overline{\Gamma}\right| \ge \delta n} (1 - 1/\overline{q})^{\theta \left|\overline{\Gamma}\right|}
\end{aligned}$$

where $\theta = \beta\lfloor \alpha n\rfloor/\lfloor \delta n\rfloor \approx \beta/q$. The number of polymer sets $\Gamma \in \mathrm{cpt}(\Gamma_X)$ with $\left|\overline{\Gamma}\right| = k$ is at most $\binom{2n}{k}\overline{q}^k$. Thus

$$\begin{aligned}\epsilon &\le \sum_{k=\lceil \delta n\rceil}^{\lfloor 2n/\theta\rfloor}\binom{2n}{k}\overline{q}^k(1-1/\overline{q})^{\theta k} \le \sum_{k=\lceil \delta n\rceil}^{\lfloor 2n/\theta\rfloor}\left(2^{H(k/2n)2n/k}\overline{q}(1-1/\overline{q})^{\theta}\right)^k\\ &\le \sum_{k=\lceil \delta n\rceil}^{\lfloor 2n/\theta\rfloor}\left((2n/k)^2\overline{q}(1-1/\overline{q})^{\theta}\right)^k,\end{aligned}$$

where the last inequality follows from Lemma 9. Let $R = (2n/\lceil \delta n\rceil)^2\overline{q}(1-1/\overline{q})^{\theta}$, then

$$\begin{aligned}\log R &< -2\log(q\alpha/2) + \log q - \beta/q^2\\ &< 2\log\frac{2}{q\alpha} + \log q - (1/\alpha - 1)/q^2\\ &< 2\log(8q^2\log q) + \log q - (4q^3\log q - 1)/q^2 < 0.\end{aligned}$$

Therefore, we have

$$\epsilon \le \sum_{k=\lceil \delta n\rceil}^{\infty} R^k = R^{\lceil \delta n\rceil}/(1-R) \le K^{-n}$$

for some $K = K(q) > 1$. □

5.3. Approximating the polymer model partition function

Lemma 31. *There is an FPTAS for $\Xi_X(1)$ for all $G \in \mathcal{G}^{\Delta}_{\alpha,\beta}$ and $X \subseteq [q]$ with $|X| \in \{\underline{q}, \overline{q}\}$.*

Proof. We use the FPTAS in Theorem 5 to design the FPTAS we need. To this end, we generate a graph $H = G^2$ in polynomial time in $|G|$ for $G \in \mathcal{G}^{\Delta}_{\alpha,\beta}$. Let H be the input to the FPTAS in Theorem 5 and we also allow the FPTAS to query the original graph G. Note that $|V(H)| = |V(G)|$. Let $\Gamma(H) = \Gamma_X(G)$ and $\Xi(H,z) = \Xi_X(z) = \sum_{\Gamma\in\mathrm{cpt}(\Gamma_{\mathcal{X}}(G))}\prod_{\gamma\in\Gamma} w(\gamma,z)$ where $w(\gamma,z) = \frac{|\mathcal{D}_\gamma|}{|X|^n(q-|X|)^n}z^{|\overline{\gamma}|}$, and

$$\left|\mathcal{D}_\gamma\right| = \left(\prod_{v\in\mathcal{L}\setminus V(\overline{\gamma})}|X\setminus Q_v|\right)\left(\prod_{v\in\mathcal{R}\setminus V(\overline{\gamma})}|([q]\setminus X)\setminus Q_v|\right),$$

and $Q_v = \omega_{\overline{\gamma}}(N_G(v)\cap V(\overline{\gamma}))$ is the set of colors for v that are occupied by its neighbors. We verify that the conditions in Theorem 5 hold:

- The degree of a monomial $\prod_{\gamma\in\Gamma} w(\gamma,z)$ is at most $|V(H)|$ since Γ is compatible.
- To compute the quantity $\left|\mathcal{D}_\gamma\right|$, we enumerate each vertex $v \in (\mathcal{L}\cup\mathcal{R})\setminus V(\overline{\gamma})$ and compute the set Q_v by enumerating the vertices adjacent to v in G. This step takes $O\left(|V(G)|\cdot\Delta\cdot\left|V(\overline{\gamma})\right|\right) \le \exp\left(O\left(\left|\overline{\gamma}\right| + \log|H|\right)\right)$ time.
- For every connected subgraph H' of H with s vertices in $\mathcal{L}$ and t vertices in $\mathcal{R}$, there are exactly $(q-|X|)^s|X|^t$ polymers γ in $\Gamma(H)$ such that $\overline{\gamma} = H'$. In addition, we can enumerate them in $O\left(q^{s+t}\right) = \exp\left(O\left(\left|H'\right|\right)\right)$ time, for q is a constant (viewed as part of the problem description).
- This one follows from Lemma 32. □

Lemma 32. *For $G \in \mathcal{G}^{\Delta}_{\alpha,\beta}$ and $X \subseteq [q]$ with $|X| \in \{\underline{q}, \overline{q}\}$, $\Xi_X(z) \ne 0$ for all $z \in \mathbb{C}$ with $|z| \le 2$.*

Proof. Let $a(\gamma) = \left|\overline{\gamma}\right|$ for $\gamma \in \Gamma_X$. We will verify that the KP-condition

$$\sum_{\gamma':\gamma'\nsim\gamma} e^{\left|\overline{\gamma'}\right|}\left|w(\gamma',z)\right| \le \left|\overline{\gamma}\right| \tag{2}$$

holds for any $\gamma \in \Gamma_X$. Let $V^* = V(\overline{\gamma}) \cup N_{G^2}(\overline{\gamma})$. Then every $\gamma' \nsim \gamma$ contains a vertex in V^* since $d_{G^2}(\overline{\gamma'},\overline{\gamma}) \le 1$. Recall that a polymer is connected in the graph G^2 which is of maximum vertex degree $\le \Delta^2$. By Lemma 7, given a vertex v, the number of connected subgraphs of G^2 containing v is at most $(e\Delta^2)^{k-1}$. Therefore, we have

$$\left|\left\{\gamma' : \gamma' \nsim \gamma \wedge \left|\overline{\gamma'}\right| = k\right\}\right| \le \left|V^*\right|(e\Delta^2)^{k-1}\overline{q}^k \le \Delta^2\left|\overline{\gamma}\right|(e\Delta^2)^{k-1}\overline{q}^k.$$

Then it holds that

$$\begin{aligned}\sum_{\gamma':\gamma'\nsim\gamma} e^{\left|\overline{\gamma'}\right|}\left|w(\gamma',z)\right| &\le \sum_{k=1}^{\lfloor\delta n\rfloor}\left|\left\{\gamma' : \gamma' \nsim \gamma \wedge \left|\overline{\gamma'}\right| = k\right\}\right| \cdot e^k \cdot \left|w(\gamma',1)\right| \cdot |z|^k \\ &\le \frac{\left|\overline{\gamma}\right|}{e}\sum_{k=1}^{\infty}\left(2e^2\Delta^2\overline{q}(1-1/\overline{q})^{\theta}\right)^k,\end{aligned}$$

where $\theta = \beta\lfloor\alpha n\rfloor/\lfloor\delta n\rfloor \approx \beta/q$ (see Lemma 30). Let $K = 2e^2\Delta^2\overline{q}(1-1/\overline{q})^{\theta}$. Then

$$\begin{aligned}\log K &< \log(2e^2\overline{q}) + 2\log\Delta - \frac{3\log e}{2q^2}\beta \\ &= \log(2e^2\overline{q}) + 2\log\left(4(\beta+1)\log(\beta+1)\right) - \frac{3\log e}{2q^2}\beta.\end{aligned}$$

Since $\beta + 1 = 1/\alpha > 4q^3\log q$, we have $\log K < -1$. Therefore,

$$\sum_{\gamma':\gamma'\nsim\gamma} e^{\left|\overline{\gamma'}\right|}\left|w(\gamma',z)\right| \le \left|\overline{\gamma}\right|\sum_{k=1}^{\infty}\frac{1}{2^k} = \left|\overline{\gamma}\right|.$$

By Lemma 6, $\Xi_X(z) \ne 0$. $\square$

5.4. Putting things together

Combining Lemma 22 and Lemma 33, we obtain our main result for counting colorings.

Theorem 3. *For $q \ge 3$ and $\Delta \ge 80q^3\log^2 q$, there is an FPTAS such that with high probability (tending to 1 as $n \to \infty$) for a graph chosen uniformly at random from $\mathcal{G}^{\mathrm{bip}}_{n,\Delta}$ the FPTAS computes the number of q-colorings of G.*

Algorithm 2 Counting colorings for $q \ge 3$ and $\Delta \ge 80q^3\log^2 q$.

1: **Input:** *A graph $G = (\mathcal{L}, \mathcal{R}, E) \in \mathcal{G}^{\Delta}_{\alpha,\beta}$ with n vertices on both partitions and $\varepsilon > 0$*
2: **Output:** *$\widehat{Z}$ such that $\exp(-\varepsilon)\widehat{Z} \le |\mathcal{C}(G)| \le \exp(\varepsilon)\widehat{Z}$*
3: **if** $n \le N$ or $\varepsilon \le 2K^{-n}$ **then**
4: Use the brute-force algorithm to compute $\widehat{Z} \leftarrow |\mathcal{C}(G)|$;
5: Exit;
6: **end if**
7: $\varepsilon' \leftarrow \varepsilon - K^{-n}$;
8: For every $X \subseteq [q]$ with $|X| \in \left\{\underline{q}, \overline{q}\right\}$, use the FPTAS in Lemma 31 to obtain $\widehat{Z_X}$, an ε'-relative approximation to $\Xi_X(1)$.
9: $\widehat{Z} \leftarrow \sum_{X\subseteq[q]:|X|\in\left\{\underline{q},\overline{q}\right\}} |X|^n(q-|X|)^n\widehat{Z_X}$.

Lemma 33. *For $q \ge 3$ and $\Delta \ge 80q^3\log^2 q$, there is an FPTAS to count the number of q-colorings for all $G \in \mathcal{G}^{\Delta}_{\alpha,\beta}$.*

Proof. First we state our algorithm. See Algorithm 2 for a pseudocode description. Fix $q \ge 3$ and $\Delta \ge 80q^3\log^2 q$. The input is a graph $G = (\mathcal{L}, \mathcal{R}, E) \in \mathcal{G}^{\Delta}_{\alpha,\beta}$ and an approximation parameter $\varepsilon > 0$. The output is a number $\widehat{Z}$ to approximate $|\mathcal{C}(G)|$. Let K_2, K_2 be the constants in Lemma 24 and Lemma 29, respectively. Let $Z = \sum_{X\subseteq[q]:|X|\in\left\{\underline{q},\overline{q}\right\}} |X|^n(q-|X|)^n\Xi_X(1)$. These two lemmas show that, since $n > N$ is sufficiently large, Z is a $K_1^{-n} + K_2^{-n} \le 2\min(K_1, K_2)^{-n} \le K^{-n}$-relative approximation to $|\mathcal{C}(G)|$ for another constant $K > 1$. If $n \le N$ or $\varepsilon \le 2K^{-n}$, we use the brute-force algorithm to compute $|\mathcal{C}(G)|$. If $\varepsilon > 2K^{-n}$, we apply the FPTAS in Lemma 31 with approximation parameter $\varepsilon' = \varepsilon - K^{-n}$ to obtain $\widehat{Z_X}$ for all $X \subseteq [q]$ with $|X| \in \left\{\underline{q}, \overline{q}\right\}$, an ε'-relative approximation to $\Xi_X(1)$. The output is $\widehat{Z} = \sum_{X\subseteq[q]:|X|\in\left\{\underline{q},\overline{q}\right\}} |X|^n(q-|X|)^n\widehat{Z_X}$. It is clear that $\exp(-\varepsilon)\widehat{Z} \le |\mathcal{C}(G)| \le \exp(\varepsilon)\widehat{Z}$.

Then we show that Algorithm 2 is indeed an FPTAS. It is required that the running time of our algorithm is bounded by $(n/\varepsilon)^{O(1)}$ for $n > N$. If $\varepsilon \le 2K^{-n}$, the running time of the algorithm would be $O(nq^n) = (n/\varepsilon)^{O(1)}$. If $\varepsilon > 2K^{-n}$, the running time of the algorithm would be $\left(n/\varepsilon'\right)^{O(1)} = \left(n/(\varepsilon - K^{-n})\right)^{O(1)} \le (2n/\varepsilon)^{O(1)} = (n/\varepsilon)^{O(1)}$. $\square$

6. Open problems

Recall that $\lambda_c(\Delta) = \frac{(\Delta-1)^{\Delta-1}}{(\Delta-2)^{\Delta}}$ is the uniqueness threshold and for $\lambda < \lambda_c(\Delta)$ there exists an FPTAS. The bound obtained in this paper $4(\log\Delta)^3/\Delta$ is larger than the uniqueness threshold. Even for the special case of $\lambda = 1$, the bound 50 is much larger than the rounding-to-integer uniqueness threshold 6. Since our technique makes use of the property that $\lambda > \lambda_c(\Delta)$, it is natural to ask whether we are able to obtain an FPTAS for $\Delta \geq 3$ and $\lambda > \lambda_c(\Delta)$ on random Δ-regular bipartite graphs. The main barrier is from the analysis of the zeros of the polymer partition function. In that (see Lemma 16), even we use some tighter ones than the KP-condition, the enumeration of connected induced subgraph is unavoidable. This requires that quantity $(1+\lambda)^{-\beta}$ beat off the quantity $(e\Delta)^2$, which is not true for values of λ slightly above the uniqueness threshold.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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